NATIONAL BUREAU OF STANDARDS REPORT

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SEQUENCING THE PURCHASE AND RETIREMENT OF FIRE ENGINES

Prepared for

The Fire Research Program
National Bureau of Standards

NBS

U.S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

NATIONAL BUREAU OF STANDARDS

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SEQUENCING THE PURCHASE AND RETIREMENT OF FIRE ENGINES

Richard Ku
Technical Analysis Division

Patsy B. Saunders
Applied Mathematics Division

Approved: ______________________
Technical Analysis Division

Approved: Alan J. Goldman
Applied Mathematics Division

U.S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

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SEQUENCING THE PURCHASE AND RETIREMENT

OF FIRE ENGINES

1. INTRODUCTION

This report describes a method to determine an "optimum" manner of sequencing the purchase and retirement of fire engines (hereafter simply called "engines"), with specific application to the Washington, D. C. Fire Department. The model developed, however, has more general applicability as regards both the equipment type and the fire department. Because of the apparent similarity of the present problem to conventional equipment replacement problems, we first review in brief some of the ideas in the equipment replacement literature.

Equipment replacement problems have a long history in industrial engineering and operations research. The reader is referred to [8] for a comprehensive bibliography on this subject. One class of equipment replacement problems balances the cost of failures against the cost of planned replacements (see [3]). If units are to operate continuously over some time period $[0, t]$ and are replaced upon failure, then typically the expected cost $C(t)$ during $[0, t]$ may be given by

$$C(t) = c_1 E[N_1(t)] + c_2 E[N_2(t)], \tag{1.1}$$

where

c_1 = per unit total cost resulting from a failure and its replacement,

c_2 = per unit total cost of replacing a non-failed item ($c_2 < c_1$),

$N_1(t)$ = the number of failures in $[0, t]$, a random variable,

$N_2(t)$ = the number of replacements of non-failed units, a random variable,

and E denotes expected value. The problem is to minimize (1.1) over the possible replacement procedures available within a given policy of replacement. Examples of replacement policies are: strictly periodic replacement, random periodic replacement and sequentially determined replacement. Electronic components typify the equipment to which this well developed mathematical theory applies.

A second class of equipment replacement problems, called "preparedness" problems, assumes that a piece of equipment is kept in a readiness state for use in case of emergency. The objective is to maintain the equipment in a state of operational readiness at minimal cost. Thus a sequence of inspection and replacement actions that minimizes the ratio of expected cost per unit time to proportion of good time, would constitute an "optimal" decision stream (see [8], [10]). Large military hardware provides examples of the type of equipment to which this class of models may be applied.

One of the basic underlying concepts of the two classes of equipment replacement models discussed so far is that of a <u>reliability</u>

function[1]. This is the probability $R(t)$ that the equipment is "good"[2] at time t (measured from a time at which the equipment is considered to be "new") and is exemplified by the negative exponential form

$$R(t) = \exp(-\lambda t). \tag{1.2}$$

A closely related concept is the failure rate, defined for any reliability function $R(t)$ as $\rho(t) = -R'(t)/R(t)$, where the prime denotes the derivative. For the negative exponential, the failure rate is the constant λ.

A third class of equipment replacement problems deals with the replacement of items that deteriorate. Mathematical models to solve this class of problems typically trade off the increasing operational and maintenance costs (and decreasing resale value) of an aging item against the cost of a new purchase, i.e., the "optimal" replacement time is that time at which these opposing forces are equalized. Dreyfus [6] used a dynamic programming approach to solve this problem under the additional complication of technological change.

The main concern of this report is the development of a model to determine purchase and retirement decisions over a planning period, subject to certain constraints, which would minimize the cost of

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[1]See [11] for a discussion of the statistical theory of reliability.

[2]It is implicitly assumed that the equipment is either in a "good" or a "failed" state.

operation of a fleet of engines during that period. The concern of the Washington, D. C. Fire Department was not with the cost of failure or the distribution of failures of fire engines per se, primarily because of the negligible number of engine failures and the inability to measure the "cost" of a single engine failure. The model developed may be regarded as an extension of the ideas represented by the third class of equipment replacement problems discussed above.

Section 2 describes a simple calculation, which serves to introduce the data at hand and compares the results of this calculation (as applied to Washington, D. C.) to those of a study [2] from which the data were obtained. A dynamic programming (DP) model is formulated and given illustrative application in Section 3, and directions for further investigation are suggested in Section 4. Appendix A develops certain details of the DP model and a listing of the DP computer code appears in Appendix B. Finally, an integer programming (IP) analog to the DP model is given in Appendix C.

2. INITIAL CONSIDERATIONS

Aside from personal communications with members of the staff of the Washington, D. C. Fire Department, the main source of data was a report by Balcolm [2]. This report also proposes a model, for determining the life-span of an engine, which will be described later.

A linear relationship between engine age and maintenance cost was used in [2], and least-squares regressions yielded three sets of coefficients, corresponding to "high usage," "medium usage," and "low usage" engines. Balcolm then obtained a "composite" equation--a weighted average (by the number of engines in the three categories)--which this report also uses. This equation is of the form:

$$u_a = U_0 + U_1 a, \tag{2.1}$$

where

a = engine age,

u_a = the maintenance cost of an engine entering its $\underline{a}^{th}$ year of service,

U_0 = 24.17,

U_1 = 122.46/year.[3]

Values of u_a are listed in Table 3.1. This relationship was adopted as the basis of the data for maintenance cost since it was felt that a more complex function could not be supported by the observed cost figures.

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[3] All monetary quantities are expressed in dollars.

A linear relationship was also used in [2] for the purchase price of a new engine, given by

$$P_t = P_0 + P_1 (t - 1900), \tag{2.2}$$

where

$P_0 = -16258.18,$

$P_1 = 576.87.$

Values of P_t are given in Table 3.1. The choice of the "base" year 1900 is not explained, but it accounts for the surprising (negative) value of P_0. The index t refers to the year for which a value of the purchase price is desired.

Using these data, a simple calculation can be made to determine an "optimum" life-span for a single engine. Assuming a zero salvage value (for simplicity)[4] and a constant purchase price, the accumulated total cost of keeping an engine for n years is

$$\begin{aligned} TC(n) &= \sum_{a=1}^{n} (U_0 + U_1 a) + P \\ &= nU_0 + U_1 \sum_{a=1}^{n} a + P \\ &= nU_0 + [n(n+1)/2]\, U_1 + P. \end{aligned} \tag{2.3}$$

Thus the <u>average</u> annual cost of keeping an engine for n years is

$$\begin{aligned} AC(n) &= TC(n)/n \\ &= U_0 + [(n+1)/2]\, U_1 + P/n. \end{aligned} \tag{2.4}$$

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[4]Constant salvage values (with respect to age) can be represented by subtracting them from U_0.

Clearly, the longer an engine is kept, the longer the time to amortize the price P, so that portion of the cost per year will decrease with n. However, the maintenance costs increase year by year. Thus, with the "optimum" life-span defined as that value of n which minimizes (2.4), the standard calculus technique of setting the derivative of (2.4) to zero and solving for n yields:

$$(d/dn)\ (AC(n)) = U_1/2 - P/n^2 = 0, \tag{2.5}$$

whence

$$n = (2P/U_1)^{1/2}. \tag{2.6}$$

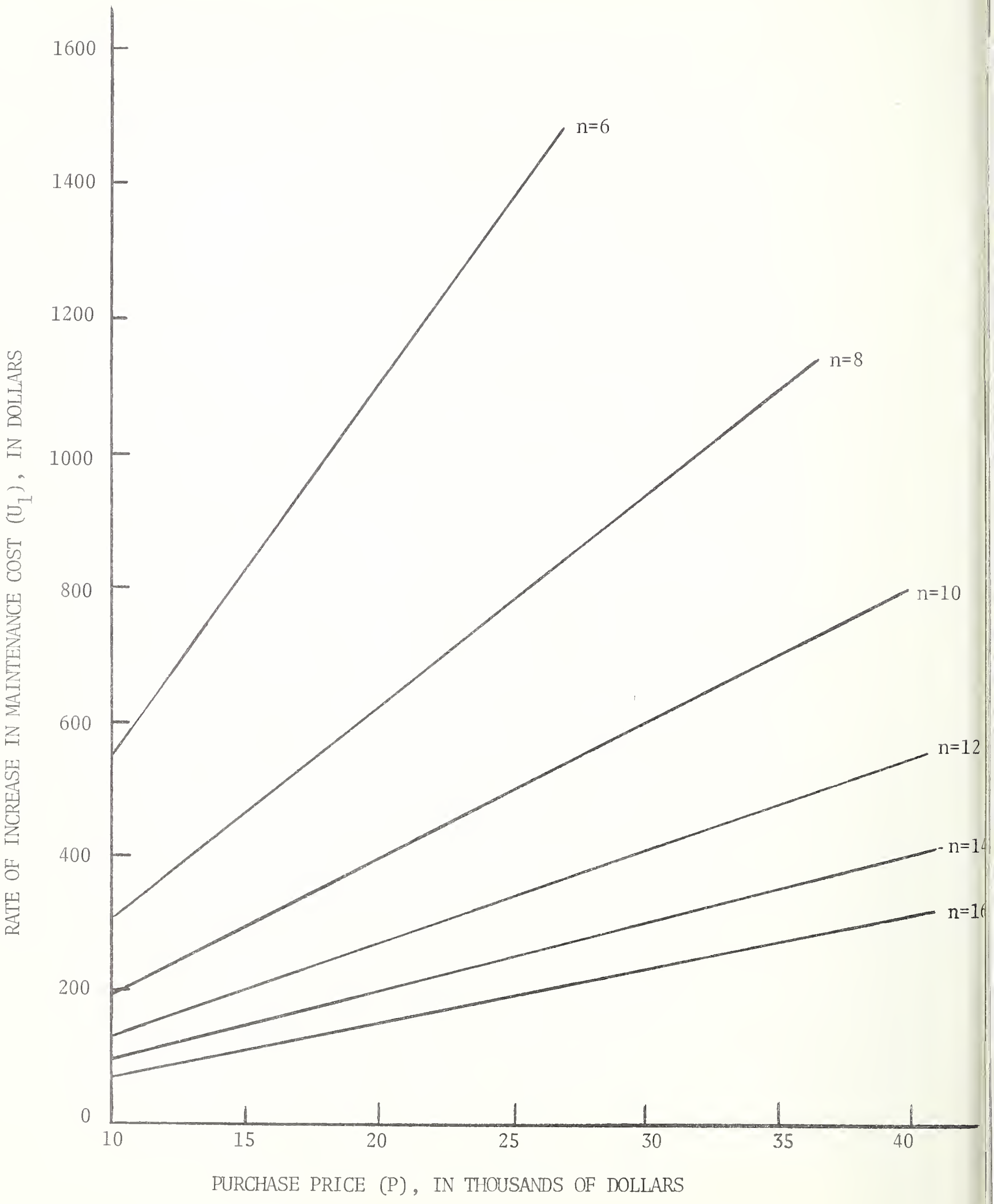
Since $P > 0$ and $n > 0$, the second derivative $2P/n^3$ is positive so that the value of n given in (2.6) ensures a minimum value of (2.4). Figure 2.1 indicates contours of the optimum value of n in the (U_1, P)-plane.

For Washington, D. C., using the 1969 purchase price, (2.6) yields $n = 19.6$, considerably larger than the present life span of 15 years. Balcolm [2] recommends a life span of 10-11 years, depending on the number of years over which an engine is linearly depreciated, using as his criterion the equality of current (resale) value and accumulated repair cost, i.e., n is chosen so that

$$P - n(P - S)/N = \sum_{a=1}^{n} U_a,$$

where N is the number of years over which an engine is depreciated and S is the salvage value of an engine after N years. (Note that Balcolm assumes that the number of years over which an engine is depreciated (N) and the number of years it is kept (n) need not

FIGURE 2.1 CONTOURS OF THE OPTIMUM ENGINE LIFE



be the same.) No rationale for this criterion is offered in [2], but the large difference between [2]'s "optimum" life span and the one derived from the present calculation indicates a significant difference between the two models.

3. A DYNAMIC PROGRAMMING MODEL

The dynamic programming (DP) model described in this section takes a somewhat different approach to the problem of equipment replacement. Instead of determining an "optimum" life-span which would be applied to all engines, the DP model begins with the existing scenario and prescribes purchasing and retiring decisions over a T-year planning horizon. (The index $t = 1, \ldots, T$ is used in this model and appropriate notation changes are made in the relevant formulas presented in Section 2.) In this sense, the model may be "tailored" to fit the initial state of affairs of any urban fire department. The reader interested in DP in general, is referred to the text [9]. For other DP formulations of equipment replacement problems, see [1] and [4].

In accordance with the concerns and objectives of the Washington, D. C. Fire Department, the DP model determines the purchases and retirements to be made during the planning horizon such that the total cost incurred during this period is minimized. The model accounts for various constraints within which a fire department must operate, e.g., constraints on the number of purchases and/or retirements which may be made in any year, the total fleet size, and the maximum allowable engine age.

The DP "state variables" (those which describe the system at each stage, or year in this case) are:

x_{1t} = the number of engines in the initial fleet which remain in year t-1,

x_{2t} = the number of new engines purchased in years 1,...,t-1,

x_{3t} = the maintenance cost in year t-1 on engines purchased in years 1,...,t-1.

(Note that $x_{1t} + x_{2t}$ is the fleet size in year t-1.) The "decision variables" are

d_{1t} = the number of engines retired from the initial fleet in year t,

d_{2t} = the number of engines purchased in year t.

It should be emphasized that retirements are made only from engines in the initial fleet, i.e., none of the engines purchased during the planning period are considered for retirement. Since the Washington, D. C. Fire Department indicated interest in a planning horizon of at most five to ten years, restriction to retiring engines from the initial fleet only is not considered a limitation.

The data required by the model are :

D_t = the minimum number of engines required during year t (checked against the fleet size after year t's decisions have been made),[5]

M_t = the maximum number of engines which may be purchased in year t,

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[5]That D_t adequately measures the demand for fire service is a simplification.

N_t = the maximum number of engines which may be retired in year t,

R = the age by which engines must be retired,

P_t = the purchase price of a new engine in year t,

Q_a = the number of a-year old engines in the initial fleet,

$m = \sum_a Q_a$ = the initial fleet size,

u_a = the maintenance cost of an engine during its $\underline{a}^{th}$ year of service,

v_{at} = the resale value in year t of an engine which was initially of age a,[6]

a_i = the age of the $\underline{i}^{th}$ youngest engine in the initial fleet (e.g., a_1 is the youngest).

As in the simple model of section 2, the maintenance costs are calculated as

$$u_a = U_0 + U_1 a,$$

with the values of U_0 and U_1, as indicated earlier. The linear relationship leads to a recursive definition of u_a,

$$\begin{aligned} u_{a+1} &= U_0 + U_1(a+1) \\ &= U_0 + U_1 a + U_1 \qquad (3.1) \\ &= u_a + U_1. \end{aligned}$$

Letting $x_t = (x_{1t}, x_{2t}, x_{3t})$, (3.1) may be used to obtain, as the stage transformation formula,

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[6]The convention is adopted that an a-year-old engine in the initial fleet enters its $(a+1)^{st}$ year of service at t=1. It is assumed, for simplicity, that decisions are made at the beginning of a year, and that $a \geq 1$.

$$x_{t+1} = (x_{1t} - d_{1t}, x_{2t} + d_{2t}, x_{3t} + u_1 d_{2t} + U_1 x_{2t}). \tag{3.2}$$

The transformation for x_{1t} and x_{2t} is clear. The value of $x_{3,t+1}$, the maintenance cost in year t on engines purchased in years 1,...,t, is obtained by adding to x_{3t} both the cost of the first year of maintenance for engines purchased in year t ($u_1 d_{2t}$), and the incremental increase in maintenance cost on engines purchased in the preceding years ($U_1 x_{2t}$), the latter deriving from (3.1).

The "stage return" is the cost of operation in year t. With the notation $d_t = (d_{1t}, d_{2t})$, the stage return is calculated as:

$$I_t(x_t, d_t) = (P_t + u_1) d_{2t} + \sum_{i=1}^{x_{1t}-d_{1t}} U_{a_i+t} - \sum_{i=x_{1t}-d_{1t}+1}^{x_{1t}} v_{a_i t} + x_{3t} + U_1 x_{2t}. \tag{3.3}$$

(footnote 7 is attached to the end of equation (3.3).)

The components of (3.3) have the following interpretations:

$(P_t + u_1) d_{2t}$ = the cost of purchasing d_{2t} engines in year t and maintaining them during the first year of service,

$\sum_{i=1}^{x_{1t}-d_{1t}} U_{a_i + t}$ = the maintenance cost in year t on engines which remain from the initial fleet,

$\sum_{i=x_{1t}-d_{1t}+1}^{x_{1t}} v_{a_i+t}$ = the revenue from retiring the d_{1t} oldest engines not previously retired,[8]

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[7] Whenever the lower limit of a summation exceeds the upper limit, the summation is taken to be zero. This is a standard notational convenience.

[8] This assumption of retiring "oldest" first" is supported by the Washington, D. C. Fire Department.

$x_{3t} + U_1 x_{2t}$ = the maintenance cost in year t on engines purchased in years $1,\ldots,t-1$.

The linear form of the maintenance cost yields the pleasing result that the values of x_{3t} are all exact multiples of U_1.[9] This, together with the fact that x_{1t} and x_{2t} are integers bounded by the constraints, makes it computationally feasible to consider all of the combinations of values that the state variables may assume in any stage. It follows that the optimal solution is exact, a condition not often found in DP problems. This characteristic is explicitly noted here as a favorable feature of the model.

The recursive equations of the DP model are:

$$f_t(x_t) = \min_{d_t} [I_t(x_t, d_t) + f_{t+1}(x_{t+1})/(1+r)],$$
$$f_T(x_T) = \min_{d_T} I_t(x_T, d_T). \tag{3.4}$$

The quantity r is a discount rate, so that division by (1+r) in the first relation of (3.4) renders $f_t(x_t)$ as the minimum present value cost of operations from years t through T, given that the state of the system in year t is x_t. Since the initial state is known to be $x_1 = (m,0,0)$, $f_1(m,0,0)$ is the optimal value of the objective, i.e., the minimum total cost of operations in years $1,\ldots,T$.

The constraints of the DP model are straightforward from the definitions of the variables and parameters:

[9] This will be proven in Appendix A.

$$0 \le d_{1t} \le N_t \quad (t = 1,\ldots,T), \tag{3.5}$$

$$0 \le d_{2t} \le M_t \quad (t = 1,\ldots,T), \tag{3.6}$$

$$x_{1t} + x_{2t} \ge D_{t-1} \quad (t = 2,\ldots,T+1), \tag{3.7}$$

$$\sum_{j=1}^{t-1} d_{ij} \ge n_t \quad (t = 2,\ldots,T+1) \tag{3.8}$$

where $n_t = \sum_{a>R-t+1} Q_a$ is the number of engines which _must_ be retired _prior to_ year t because of the age limitation R. Note that by definition the initial conditions are: $x_{11} = m$, $x_{21} = 0$, $x_{31} = 0$, and $n_1 = 0$. With the definition $D_0 = m$, (3.7) and (3.8) automatically hold for t = 1.

The constraints (3.5) - (3.8) and the relationships among the state and decision variables lead to interesting and computationally useful results which are detailed in Appendix A. Suffice it to say here that a special computer code,[10] developed as a part of this effort, takes advantage of these results to make it possible to solve larger problems than could be handled by a general purpose DP code. Furthermore, experience thus far has indicated that computer running times are significantly shorter using the special code. For example, one of the runs to be discussed below took 12 seconds using the special code, while the general purpose code[11] took 227 seconds.

[10] A listing of this code appears in Appendix B.

[11] This code is an extension of the code documented in [5].

(Both codes are written in FORTRAN V and runs were made on the UNIVAC 1108 at NBS under the EXEC II Operating System.)

In exercising the DP model, the maintenance costs and purchase prices were the same as those discussed previously (cf., Section 2). The purchase price function was modified to

$$P_t = P_0 + P_1 \; (70 + t), \tag{3.9}$$

so that t = 1 would correspond to 1971. The values of P_0 and P_1 are unaffected by the modification and remain as listed under equation (2.2). The resale values v_{at} were calculated on the basis of (3.9), assuming an annual depreciation rate ρ, as

$$v_{at} = (1 - \rho)^{a+t-1} \; [P_0 + P_1 \; (70-a+1)], \tag{3.10}$$

so that resale values of engines in the initial fleet (purchased prior to t = 1) could be calculated from the appropriate purchase prices.[12] Finally, values of Q_a were obtained directly from the Washington, D. C. Fire Department's inventory of engines. These data are given in Table 3.1 with T = 5 (a five-year planning horizon).[13]

For the remaining data specifications, it was suggested by members of the Fire Department staff to take R = 15 (the present maximum engine age in Washington), D_t = 64 for t = 0,...,5 (i.e. constant

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[12] A geometric depreciation is not required by the model. It is incorporated in the code, but can easily be modified with minor coding changes.

[13] Members of Fire Department staff advised that a planning period of more than five years is unreasonable.

TABLE 3.1 - DATA FOR THE DYNAMIC PROGRAMMING MODEL

a	Q_a	u_a*	v_{at}*				
			t=1	t=2	t=3	t=4	t=5
1	4	147	14474	8684	5211	3126	1876
2	0	269	8477	5086	3052	1831	1099
3	10	392	4961	2977	1786	1072	643
4	5	514	2902	1741	1045	627	376
5	0	636	1696	1018	611	366	220
6	5	759	991	595	357	214	128
7	10	881	578	347	208	125	75
8	0	1004	337	202	121	73	44
9	5	1126	197	118	71	42	25
10	5	1249	114	69	41	25	15
11	3	1371	67	40	24	14	9
12	4	1494	39	23	14	8	--
13	4	1616	22	13	8	--	--
14	4	1739	13	8	--	--	--
15	5	1861	8	--	--	--	--

t	P_t*
1 (1971)	24700
2	25276
3	25853
4	26430
5 (1975)	27007

*Values have been rounded to the nearest dollar.

minimum required fleet size equal to the present fleet size), and $M_t = N_t = 6$ for $t = 1, \ldots, 5$ (constant and equal purchase and retirement ceilings).

A base run was made with no discounting, i.e., $r = 0$, and the resultant "optimal" decisions were to purchase and retire 6 engines in each of the first three years and to purchase and retire 2 engines in year 4, i.e., $d_{1t} = d_{2t} = 6$ (t=1, 2, 3), $d_{14} = d_{24} = 2$, $d_{15} = d_{25} = 0$. Note from the age distribution Q_a in Table 3.1 that 20 engines reach the mandatory retirement age by year 5 (i.e., $n_6 = 20$). Since the maximum number of retirements permissible is 6 in each year, the optimal policy is to retire the 20 engines as soon as possible (ASAP policy), replacing them with new engines to meet the minimum required fleet size.

The above results are not surprising in view of the discount rate $r = 0$. Increasing maintenance costs, decreasing salvage values, and increasing purchase prices all indicate early retirement. The same policy is optimal in the extreme case where the purchase price is always zero. It is intuitively obvious that in this situation the ASAP policy is optimal regardless of the value of r, since the newly acquired (free) engines are operated at a lower maintenance cost than are the old ones.

In order to study the effect of the discount rate r on the optimal decisions, a series of runs was made with U_1 as a parameter, taken from 62.46 to 162.46 in increments of 10.00. [Recall that the "nominal" value of U_1 is 122.46.] Initially, r was varied from

0.0 to 0.5 in increments of 0.1 (a very rough grid), and based upon these results, smaller ranges with finer increments were studied for certain values of U_1. The following observations were made consistently from the outputs of all the runs:

(1) The only engines retired were the 20 which reach their maximum age during the 5-year planning period.

(2) In every year, the numbers of purchases and retirements were the same. This may be attributable to the constant demand and to the constant and equal values of M_t and N_t over all t.

(3) For those values of r considered, there was a value r_E such that for $r \leq r_E$ the ASAP policy was optimal, and a value r_L such that for $r \geq r_L$ the optimal policy was to retire as late as possible (ALAP policy) [The ALAP policy has $d_{11} = d_{12} = 5$, $d_{1t} = d_{2t} = 4$ (t=2, 3, 4), $d_{15} = d_{25} = 3$ for this particular problem.]

(4) The values of r_E, r_L and $r_L - r_E$ are monotonically increasing functions of U_1.

The values of U_1 for which the behavior of the optimal policy, as a function of r, was studied in greater detail are listed in Table 3.2 together with the relevant results. All other values of U_1 considered gave rise to values of $r_E = 0.0$ and $r_L = 0.1$ in the initial runs. It can be seen from Table 3.2 that the finest

TABLE 3.2 - RESULTS OF FINER VARIATION OF r FOR CERTAIN VALUES OF THE PARAMETER U_1

U_1	Range of r	Increment	r_E	r_L
62.46	.01 - .10	.01	.05	.06
122.46	.08 - .09	.001	.080	.089
152.46	.01 - .20	.01	.09	.11
162.46	.01 - .20	.01	.10	.1[illegible]

analysis with the smallest increments for r was made for the "nominal" value of $U_1 = 122.46$. For $.080 < r < .089$ the optimal decisions were "mixed", i.e., neither an ASAP nor an ALAP policy. For example with $r = .085$, the optimal decisions were

$$d_{11} = d_{12} = 5, \qquad d_{12} = d_{22} = 6,$$

$$d_{13} = d_{23} = 6, \qquad d_{14} = d_{24} = 3,$$

$$d_{15} = d_{25} = 0.$$

The "critical" range of r (.080, .089) is quite small, but it should be noted that the values $M_t = N_t = 6$ do not permit a drastic difference between the ASAP policy and the ALAP policy.

It is clear that if a value of r is specified, then the DP model may be run to determine the optimal policy. If r cannot be specified, then the values of r_E and r_L may be determined for a given value of U_1. Then one need only specify whether $r \leq r_E$ or $r \geq r_L$ to conclude that the ASAP policy or ALAP policy, respectively, is optimal.

One run was made with $M_t = N_t = 10$ for all t and the other data remaining the same. With $r = 0$, the ASAP policy resulted; in this case $d_{11} = d_{12} = d_{21} = d_{22} = 10$, $d_{1t} = d_{2t} = 0$ $(t = 3, 4, 5)$. Unfortunately, lack of time prevented further study of this case. Intuitively, one might expect a greater "critical" range of r since the larger values of M_t and N_t given rise to a greater difference between the ASAP and ALAP policies.

4. CONCLUDING COMMENTS

It should be emphasized that the DP model has considerably greater generality than was indicated in the limited application to Washington, D. C. The only model constraint on the data is that they be self-consistent (e.g., M_t and N_t must be consistent with D_t). If, for example, an urban fire department sees fit to reduce its fleet size because of overkill capacity or perhaps because of declining demand, and the values of M_t and N_t fluctuate because of a fluctuating budget, then a greater portion of the model's generality could be exploited. The interactions among the variables and parameters of the model which are evident in Appendix A should support this contention.

On the other hand, time limitations prevented any attempts to examine the model with particular relationships among the parameters. It seems reasonable that certain conditions, e.g., $M_t = N_t =$ constant, or D_t = a constant for all t, could lead perhaps to closed-form optimal solutions, or at least might simplify the necessary DP calculations. Further research along these lines is recommended. In addition to these basic issues, there is a need for further sensitivity tests, with respect to the discount rate and the value of U_1, for other values of the parameters M_t, N_t, D_t, and R. For instance, the optimal values of the objective f_1 (m, 0, 0) could be compared for different values of R (in some reasonable range of maximum ages), leading to an "optimal"

value of R (i.e. one which minimizes f_1 (m, 0, 0)). Finally, runs with depreciation rate ρ varying, or using a different (perhaps linear) depreciation policy, would be desirable.

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APPENDIX A

DETAILS OF THE DYNAMIC PROGRAMMING MODEL

This Appendix develops certain details of the DP model described in Section 3. In particular, relationships among the variables are investigated which make it possible to examine a limited number of states and decisions for which the stage returns $I_t(x_t, d_t)$ are calculated. Although technical in nature, this aspect of the problem is of great importance to computational feasibility in the sense that computer storage requirements and running times depend on the number of states and decisions the algorithm must consider.

The definitions of the relevant variables and parameters are repeated below for the reader's convenience:

x_{1t} = the number of engines remaining from the initial fleet in year t-1 (t=1, ..., T+1),

x_{2t} = the number of new engines purchased in years 1, ..., t-1 (t=1, ..., T+1),

x_{3t} = the maintenance cost during year t-1 on engines purchased in years 1, ..., t-1 (t=1,...,T+1),

d_{1t} = the number of engines retired in year t (t=1,...,T),

d_{2t} = the number of engines purchased in year t (t=1,...,T),

D_t = the minimum number of engines required in year t (t=1,...,T),

M_t = the maximum number of engines which may be purchased in year t (t=1,...,T),

N_t = the maximum number of engines which may be retired in year t (t=1,...,T),

R = the age by which engines <u>must</u> be retired,

Q_a = the number of <u>a</u>-year-old engines in the initial fleet.

From these definitions, we may calculate two other quantities which are used throughout the sequel:

$m = \sum_{a} Q_a$ = the number of engines in the initial fleet,

$n_t = \sum_{a>R-t+1} Q_a$ = the number of engines which must be retired prior to year t because of the age limitation R$(t=2,\ldots T+1)$.

Note that by definition: $x_{11} = m$, $x_{21} = 0$, $x_{31} = 0$, and $n_1 = 0$.

It is notationally convenient to adopt the convention $D_0 = m$.

Using the definitions above, we may immediately establish the relationships

$$x_{1t} = x_{1,t-1} - d_{1,t-1} \quad (t=2,\ldots,T+1) \tag{A-1}$$

$$x_{2t} = x_{2,t-1} + d_{2,t-1} \quad (t=2,\ldots,T+1) \tag{A-2}$$

$$0 \leq d_{1t} \leq N_t \quad (t=1,\ldots,T) \tag{A-3}$$

$$0 \leq d_{2t} \leq M_t \quad (t=1,\ldots,T) \tag{A-4}$$

$$x_{1t} + x_{2t} \geq D_{t-1} \quad (t = 1,\ldots,T+1) \tag{A-5}$$

$$\sum_{j=1}^{t-1} d_{1j} \geq n_t \quad (t=1,\ldots,T+1) \tag{A-6}$$

We maintain our convention regarding sums, viz., a sum is zero if its lower limit exceeds its upper limit. For example, (A-6) is valid for t=1 since both sides of the inequality are zero. The variations in the index-ranges are due to the fact that the state

variables refer to the system upon entering year t (or leaving year t-1), while the decision variables refer to decisions made in year t (presumed to be made at the beginning of year t). Note that x_{3t} does not appear in (A-1) - (A-6). This is because x_{3t} depends only upon the distribution of the purchases x_{2t} over the years 1,...,t-1. This observation is discussed at greater length subsequently.

It is clear that the stream of decisions $d_{2t} = M_t$ (t=1,...,T) and the resultant stream of states $x_{2t} = \sum_{j=1}^{t-1} M_j$ (t=1,...,T) do not violate (A-1) - (A-6). Hence the least upper bound (LUB) of d_{2t} is

$$\mu(d_{2t}) = M_t \quad (t=1,...,T), \tag{A-7}$$

and the LUB of x_{2t} is

$$\mu(x_{2t}) = \sum_{j=1}^{t-1} M_j \quad (t=2,...,T). \tag{A-8}$$

[Recall that $x_{21} = 0$ by definition.] We use (A-7) and (A-8) to develop the LUB and the greatest lower bound (GLB) of x_{1t} (t=2,...,T+1). [Recall that $x_{11} = m$ by definition.]

For a lower bound on x_{1t}, we observe first that for $t \leq \tau \leq T+1$, $x_{1t} \geq x_{1\tau}$, so that

$$x_{1t} + \sum_{j=1}^{\tau-1} M_j \geq x_{1\tau} + \sum_{j=1}^{\tau-1} d_{2j} = x_{1\tau} + x_{2\tau} \geq D_{\tau-1},$$

implying that

$$x_{1t} \geq D_{\tau-1} - \sum_{j=1}^{\tau-1} M_j \qquad (t \leq \tau \leq T + 1). \tag{A-9}$$

For $1 \leq \tau < t$, we have

$$x_{1\tau} = x_{1t} + \sum_{j=\tau}^{t-1} d_{1j},$$

so that

$$x_{1t} + \sum_{j=1}^{\tau-1} M_j \geq x_{1\tau} - \sum_{j=\tau}^{t-1} d_{1j} + \sum_{j=1}^{\tau-1} d_{2j}$$

$$\geq x_{1\tau} - \sum_{j=\tau}^{t-1} N_j + x_{2\tau}$$

$$\geq D_{\tau-1} - \sum_{j=\tau}^{t-1} N_j,$$

implying that

$$x_{1t} \geq D_{\tau-1} - \sum_{j=1}^{\tau-1} M_j - \sum_{j=\tau}^{t-1} N_j \qquad (1 \leq \tau < t). \tag{A-10}$$

With our convention concerning sums, (A-9) and (A-10) can be combined as

$$x_{1t} \geq \max_{1 \leq \tau \leq T+1} [D_{\tau-1} - \sum_{j=1}^{\tau-1} M_j - \sum_{j=\tau}^{t-1} N_j] \quad (t= 2,\ldots,T+1),$$

where the case $\tau=1$ corresponds to the condition $x_{1t} \geq m - \sum_{j=1}^{t-1} N_j$.

We also require $x_{1t} \geq 0$. Hence

$$x_{1t} \geq \max \{0, \max_{1 \leq \tau \leq T+1} [D_{\tau-1} - \sum_{j=1}^{\tau-1} M_j - \sum_{j=\tau}^{t-1} N_j]\} \quad (t=2,\ldots,T+1). \tag{A-11}$$

For an upper bound to x_{1t}, we note that for $t \leq \tau \leq T + 1$,

$n_\tau \leq \sum_{j=1}^{\tau-1} d_{1j} \leq \sum_{j=1}^{t-1} d_{1j} + \sum_{j=t}^{\tau-1} N_j = m - x_{1t} + \sum_{j=t}^{\tau-1} N_j$, so that

$$x_{1t} \leq m - \max_{t \leq \tau \leq T+1} [n_\tau - \sum_{j=t}^{\tau-1} N_j] \quad (t=2,\ldots,T+1), \tag{A-12}$$

where the case $\tau=t$ corresponds to the condition $x_{1t} \leq m-n_t$.

We now let

$$\lambda(x_{1t}) = \max \{0, \max_{1 \leq \tau \leq T+1} [D_{\tau-1} - \sum_{j=1}^{\tau-1} M_j - \sum_{j=\tau}^{t-1} N_j]\} \quad (t=2,\ldots,T+1), \tag{A-13}$$

$$\mu(x_{1t}) = m - \max_{t \leq \tau \leq T+1} [n_\tau - \sum_{j=t}^{\tau-1} N_j] \quad (t=2,\ldots,T+1), \tag{A-14}$$

and we show that the formulas (A-13) and (A-14) give the GLB and LUB of x_{1t}, respectively. This is accomplished by showing that the $\lambda(x_{1t})$ $(t=1,\ldots,T+1)$ and $\mu(x_{1t})$ $(t=1,\ldots,T+1)$ are feasible streams of the state variables x_{1t}. From (A-13) we know that

$$\lambda(x_{1t}) \geq D_{t-1} - \sum_{j=1}^{t-1} M_j, \text{ or}$$

$$\lambda(x_{1t}) + \sum_{j=1}^{t-1} M_j \geq D_{t-1}. \tag{A-15}$$

Since $\lambda(x_{1t}) \leq \mu(x_{1t})$ must hold for the problem to be feasible, it follows that

$$\mu(x_{1t}) + \sum_{j=1}^{t-1} M_j \geq D_{t-1}. \tag{A-16}$$

Next, (A-14) implies

$$\mu(x_{1t}) \leq m - n_t, \tag{A-17}$$

from which it follows that

$$\lambda(x_{1t}) \leq m - n_t. \tag{A-18}$$

Relations (A-14) and (A-18) imply, respectively, that demand is met in year t-1 with state $\lambda(x_{1t})$ and that required retirements are met with state $\lambda(x_{1t})$. Relations (A-16) and (A-17) imply that these same two conditions are met by the state $\mu(x_{1t})$.

We now state an obvious fact.

Lemma 1. If $\{a_i\}$ and $\{b_i\}$ are finite sequences and k_1 and k_2 are constants such that $k_1 \leq a_i - b_i \leq k_2$ for all i, then $k_1 \leq \max_i a_i - \max_i b_i \leq k_2$.

In order to show that $\lambda(x_{1t})$ and $\mu(x_{1t})$ are feasible streams, it remains only to show the following two propositions.

Proposition 1. $0 \leq \lambda(x_{1t}) - \lambda(x_{1,t+1}) \leq N_t \quad (t=1,\ldots,T+1)$.

Proof. For arbitrary t, let $a_0 = b_0 = 0$, and let

$$a_\tau = D_{\tau-1} - \sum_{j=1}^{\tau-1} M_j - \sum_{j=\tau}^{t-1} N_j \quad (\tau=1,\ldots,T+1),$$

$$b_\tau = D_{\tau-1} - \sum_{j=1}^{\tau-1} M_j - \sum_{j=\tau}^{t} N_j \quad (\tau=1,\ldots,T+1).$$

It is clear that $0 \leq a_\tau - b_\tau \leq N_t$ $(\tau=0,\ldots,T+1)$, so that Lemma 1 implies $0 \leq \max_{0\leq\tau\leq T+1} a_\tau - \max_{0\leq\tau\leq T+1} b_\tau \leq N_t$, or $0 \leq \lambda(x_{1t}) - \lambda(x_{1,t+1}) \leq N_t$, as stated.

Proposition 2. $0 \leq \mu(x_{1t}) - \mu(x_{1,t+1}) \leq N_t \quad (t=1,\ldots,T+1).$

Proof. For arbitrary t, let $a_{t+1} = n_{t+1}$, $b_{t+1} = \max [n_t, n_{t+1} - N_t]$,

$$a_\tau = n_\tau - \sum_{j=t+1}^{\tau-1} N_j \quad (\tau=t+2,\ldots,T+1),$$

$$b_\tau = n_\tau - \sum_{j=t+1}^{\tau-1} - N_t \quad (\tau=t+2,\ldots,T+1).$$

For $\tau=t+2,\ldots,T+1$, it is clear that $0 \leq a_\tau - b_\tau \leq N_t$. If $b_{t+1} = n_t$, then $a_{t+1} - b_{t+1} = n_{t+1} - n_t \geq 0$ by definition, and in this case $n_t \geq n_{t+1} - N_t$, so that $a_{t+1} - b_{t+1} = n_{t+1} - n_t \leq N_t$. If $b_{t+1} = n_{t+1} - N_t$, then $a_{t+1} - b_{t+1} = n_{t+1} - (n_{t+1} - N_t) = N_t \geq 0$. Hence $0 \leq a_\tau - b_\tau \leq N_t$ $(\tau=t+1,\ldots,T+1)$, so that Lemma 1 implies $0 \leq \max_\tau a_\tau - \max_\tau b_\tau \leq N_t$. Therefore ,

$$0 \leq (m - \max_\tau b_\tau) - (m - \max_\tau a_\tau) = \mu(x_{1t}) - \mu(x_{1,t+1}) \leq N_t.$$

Propositions 1 and 2 imply that $\lambda(x_{1,t+1})$ can be "reached" from $\lambda(x_{1t})$ with a feasible decision, and that $\mu(x_{1,t+1})$ can be "reached" from $\mu(x_{1t})$ with a feasible decision, respectively. [See (A-3).] These propositions together with the conclusions drawn from (A-15) - (A-18) imply that the $\lambda(x_{1t})$ and the $\mu(x_{1t})$ are feasible streams, and this together with (A-11) and (A-12) in turn imply that $\lambda(x_{1t})$ and $\mu(x_{1t})$ are the GLB and the LUB of x_{1t}, respectively.

Fix a value of x_{1t}, say $\hat{x}_{1t}$, with $\lambda(x_{1t}) \leq \hat{x}_{1t} \leq \mu(x_{1t})$. We now develop the GLB of x_{2t}, given $\hat{x}_{1t}$. For $t < \tau \leq T+1$, we have $x_{1\tau} \leq \mu(x_{1\tau})$ and $x_{1\tau} \leq \hat{x}_{1t}$, and so $x_{1\tau} \leq \min\,[\hat{x}_{1t}, \mu(x_{1\tau})]$. Hence,

$$\min[\hat{x}_{1t}, \mu(x_{1\tau})] + x_{2t} + \sum_{j=t}^{\tau-1} M_j \geq x_{1\tau} + x_{2\tau} \geq D_{\tau-1},$$

so

$$x_{2t} \geq \max_{t<\tau\leq T+1} \{D_{\tau-1} - \sum_{j=t}^{\tau-1} M_j - \min\,[\hat{x}_{1t}, \mu(x_{1\tau})]\}. \tag{A-19}$$

For $1 \leq \tau \leq t$, $x_{1\tau} \leq \mu(x_{1\tau})$ and $x_{1\tau} = \hat{x}_{1t} + \sum_{j=\tau}^{t-1} d_{1j} \leq \hat{x}_{1t} + \sum_{j=\tau}^{t-1} N_j$. Thus $x_{1\tau} \leq \min[\mu(x_{1\tau}), \hat{x}_{1t} + \sum_{j=\tau}^{t-1} N_j] \triangleq x^*_{1\tau}$. [The notation "$\triangleq$" means "defined as."] Essentially $x^*_{1\tau}$ is the largest value of $x_{1\tau}$ such that $\hat{x}_{1t}$ can be "reached" from it by feasible decisions.

Now, for $\tau \le \sigma \le t$

$$x^*_{1\sigma} + x_{2\tau} + \sum_{j=\tau}^{\sigma-1} M_j \ge x^*_{1\sigma} + x_{2\sigma} \ge D_{\sigma-1}.$$

Hence $x_{2\tau} \ge D_{\sigma-1} - x^*_{1\sigma} - \sum_{j=\tau}^{\sigma-1} M_j$. For $1 \le \sigma \le \tau$,

$$x^*_{1\sigma} + x_{2\tau} \ge x^*_{1\sigma} + x_{2\sigma} \ge D_{\sigma-1},$$

implying $x_{2\tau} \ge D_{\sigma-1} - x^*_{1\sigma}$. Again, using our convention regarding sums, we have

$$x_{2\tau} \ge \max_{1\le\sigma\le t} [D_{\sigma-1} - x^*_{1\sigma} - \sum_{j=\tau}^{\sigma-1} M_j] \triangleq x^*_{2\tau} \quad (1\le\tau\le t), \tag{A-20}$$

and in particular

$$x_{2t} \ge \max_{1\le\tau\le t} [D_{\sigma-1} - x^*_{1\sigma}] = \max_{1\le\tau\le t} \{D_{\tau-1} - \min[\mu(x_{1\tau}), \hat{x}_{1t} + \sum_{j=\tau}^{t-1} N_j]\}. \tag{A-21}$$

Let

$$\lambda(x_{2t}; \hat{x}_{1t}) = \max_{1\le\tau\le T+1} \{D_{\tau-1} - \sum_{j=t}^{\tau-1} M_j - \min[\mu(x_{1\tau}), \hat{x}_{1t} + \sum_{j=\tau}^{t-1} N_j]\}. \tag{A-22}$$

For $1\le\tau\le t$ (A-22) reduces to (A-21), and for $t<\tau\le T+1$ (A-22) reduces to (A-19), so that $\lambda(x_{2t}; \hat{x}_{1t})$ is a lower bound on x_{2t}, given $\hat{x}_{1t}$. We show that $\lambda(x_{2t}; \hat{x}_{1t})$ is the GLB of x_{2t}, given $\hat{x}_{1t}$, by first showing the existence of a feasible stream <u>to</u> $\lambda(x_{2t}; \hat{x}_{1t})$ and then showing that this state can be completed into the <u>future</u> with a stream feasible in years τ for $t<\tau\le T+1$. It is easily shown that the feasibility condition $\lambda(x_{2t}; \lambda(x_{1t})) \le \mu(x_{2t})$ follows from the feasibility condition $\lambda(x_{1t}) \le \mu(x_{1t})$.

First, note that $(x^*_{1t}, x^*_{2t}) = (\hat{x}_{1t}, \lambda(x_{2t}; \hat{x}_{1t}))$. We show that the sequence $\{(x^*_{1\tau}, x^*_{2\tau})\}$ $(\tau = 1,\ldots,t)$ is the desired stream. That $x^*_{1\tau} \leq \mu(x_{1\tau})$ follows from the definition of $x^*_{1\tau}$. Since $\mu(x_{1\tau}) \geq \lambda(x_{1\tau})$ and

$$\hat{x}_{1t} + \sum_{j=\tau}^{t-1} N_j \geq \lambda(x_{1t}) + \sum_{j=\tau}^{t-1} [\lambda(x_{1j}) - \lambda(x_{1,j+1})] = \lambda(x_{1\tau}),$$

we have $x^*_{1\tau} \geq \lambda(x_{1\tau})$. (The inequality in the above expression follows from $\hat{x}_{1t} \geq \lambda(x_{1t})$ and Proposition 1.) Therefore, $\lambda(x_{1\tau}) \leq x^*_{1\tau} \leq \mu(x_{1\tau})$.

Next we observe that $0 \leq \mu(x_{1\tau}) - \mu(x_{1,\tau+1}) \leq N_t$, by Proposition 2 and that $\hat{x}_{1t} + \sum_{j=\tau}^{t-1} N_j - (\hat{x}_{1t} + \sum_{j=\tau+1}^{t-1} N_j) = N_\tau$. It follows for all combinations of cases for $x^*_{1\tau}$ and $x^*_{1,\tau+1}$, that $0 \leq x^*_{1\tau} - x^*_{1,\tau+1} \leq N_\tau$. We have already seen that $x^*_{2\tau} \geq D_{\tau-1} - x^*_{1\tau}$, so that $x^*_{1\tau} + x^*_{2\tau} \geq D_{\tau-1}$. Thus far we have shown that the sequence $\{x^*_{1\tau}\}$ $(\tau=1,\ldots,t)$ is feasible. To complete the proof for $x^*_{2\tau}$, it remains to show that $0 \leq x^*_{2,\tau+1} - x^*_{2\tau} \leq M_\tau$. This result follows from Lemma 1 with $a_\sigma = D_{\sigma-1} - x^*_{1\sigma} - \sum_{j=\tau+1}^{\sigma-1} M_j$, $b_\sigma = D_{\sigma-1} - x^*_{1\sigma} - \sum_{j=\tau}^{\sigma-1} M_j$, since $a_\sigma - b_\sigma = M_\tau$ for $\tau+1 \leq \sigma \leq t$ and $a_\sigma - b_\sigma = 0$ for $1 \leq \sigma \leq \tau$.

To show that $(\hat{x}_{1t}, \lambda(x_{2t}; \hat{x}_{1t}))$ can be completed into the future to a stream feasible in years $t < \tau \leq T+1$, we choose $d_{2\tau} = M_\tau$ and choose $d_{1\tau}$ so that $x_{1\tau} = \min[\hat{x}_{1t}, \mu(x_{1\tau})]$. Note that the sequence $\{x_{1\tau}\}$ $(\tau = t+1,\ldots,T+1)$ is non-increasing. That the condition $x_{1\tau} + x_{2\tau} \geq D_{\tau-1}$ holds is a direct consequence of the way $\lambda(x_{2t}; \hat{x}_{1t})$ was derived. Next, $x_{1\tau} \leq \mu(x_{1\tau}) \leq m-n_\tau$, so that the required number of retirements is met. We need only show that $0 \leq x_{1\tau} - x_{1,\tau+1} \leq N_\tau$ $(t<\tau\leq T)$ to complete the proof. The left-hand inequality is clear from Proposition 2. For the right-hand inequality, if $x_{1\tau} = \mu(x_{1\tau})$, then $\mu(x_{1,\tau+1}) \leq \mu(x_{1\tau}) \leq \hat{x}_{1t}$, so that $x_{1\tau} - x_{1,\tau+1} \leq \mu(x_{1\tau}) - \mu(x_{1,\tau+1}) \leq N_t$ by Proposition 2. If $x_{1\tau} = x_{1,\tau+1} = \hat{x}_{1t}$, then the result is clear. If $x_{1\tau} = \hat{x}_{1t}$ and $x_{1,\tau+1} = \mu(x_{1,\tau+1})$, then $x_{1\tau} - x_{1,\tau+1} = \hat{x}_{1t} - \mu(x_{1,\tau+1}) \leq \mu(x_{1\tau}) - \mu(x_{1,\tau+1}) \leq N_t$, again by Proposition 2.

We now assume given a state $\hat{x}_{1t}$, $\lambda(x_{1t}) \leq \hat{x}_{1t} \leq \mu(x_{1t})$, and a state x_{2t}, $\lambda(x_{2t}; \hat{x}_{1t}) \leq \hat{x}_{2t} \leq \mu(x_{2t})$, and we derive bounds on x_{3t} $(2\leq t\leq T)$. [Recall that $x_{31} = 0$ by definition.] For the given states $\hat{x}_{1t}$, $\hat{x}_{2t}$ these bounds $\lambda(x_{3t}; \hat{x}_{1t}, \hat{x}_{2t})$ and $\mu(x_{3t}; \hat{x}_{1t}, \hat{x}_{2t})$ correspond to a "purchase late" scenario and a "purchase early" scenario, respectively, i.e. the smallest value of x_{3t} is realized when the $\hat{x}_{2t}$ engines are purchased as close to t as possible,

while the largest value of x_{3t} is realized when the $\hat{x}_{2t}$ engines are purchased as distant from t as possible. These intuitive concepts are formulated mathematically in the following paragraphs.

For the "purchase late" scenario, we observe that

$\hat{x}_{2t} - \sum_{j=\tau}^{t-1} M_j$ is the smallest value of $x_{2\tau}$ which can "reach"

$\hat{x}_{2t}$. Hence $x_{2\tau} \geq \hat{x}_{1t} - \sum_{j=\tau}^{t-1} M_j$. We also have $x_{2\tau} \geq x^*_{2\tau}$, as derived above. Combining these we have

$$x_{2\tau} \geq \max[x^*_{2\tau}, \hat{x}_{1t} - \sum_{j=\tau}^{t-1} M_j] \triangleq \bar{x}_{2\tau} \qquad (2 \leq \tau \leq t).$$

To show that the $\bar{x}_{2\tau}$ correspond to the GLB of x_{3t}, given $\hat{x}_{1t}$ and $\hat{x}_{2t}$, it suffices to show that the sequence $\{(x^*_{1\tau}, \bar{x}_{2\tau})\}$ $(\tau=1,\ldots,t)$ is feasible. We have already shown that $x^*_{1\tau}$ is in the appropriate range and that $0 \leq x^*_{1t} - x^*_{1,\tau+1} \leq N_\tau$. That demand is met follows from $x^*_{1\tau} + \bar{x}_{2\tau} \geq x^*_{1\tau} + x^*_{2\tau} \geq D_{\tau-1}$. Finally, $0 \leq \bar{x}_{2,\tau+1} - \bar{x}_{2\tau} \leq M_\tau$ follows from $0 \leq x^*_{2,\tau+1} - x^*_{2\tau} \leq M_\tau$ and $\hat{x}_{2t} - \sum_{j=\tau+1}^{t-1} M_j - (\hat{x}_{2t} - \sum_{j=\tau}^{t-1}) = M_\tau$, for all combinations of cases for $\bar{x}_{2\tau}$ and $\bar{x}_{2,\tau+1}$. The number of purchases made in year τ in the "purchase late" scenario is

$\bar{x}_{2,\tau+1} - \bar{x}_{2\tau}$, so that

$$\lambda(x_{3t};\hat{x}_{1t},\hat{x}_{2t}) = \sum_{\tau=1}^{t-1} u_{t-\tau}(\bar{x}_{2,\tau+1} - \bar{x}_{2\tau}) \quad (t-2,\ldots,T). \tag{A-23}$$

The "purchase early" scenario is somewhat simpler. We have

$\hat{x}_{2t} \leq \sum_{j=1}^{t-1} M_j$, so there exits a largest $\tau(2\leq\tau\leq t)$, say $\tau=\sigma$,

for which $\hat{x}_{2t} \geq \sum_{j=1}^{\sigma-1} M_j$. Let $\tilde{x}_{2\tau} = \sum_{j=1}^{\tau-1} M_j$ for $1\leq\tau<\sigma$ and $\tilde{x}_{2\tau} = \hat{x}_{2t}$

for $\sigma\leq\tau\leq t$. The sequence $\{\tilde{x}_{2\tau}\}$ $(\tau=1,\ldots,t-1)$ is clearly feasible.

Hence

$$\mu(x_{3t};\hat{x}_{1t},\hat{x}_{2t}) = \sum_{\tau=1}^{t-1} u_{t-\tau}(\tilde{x}_{2,\tau+1} - \tilde{x}_{2\tau}) \quad (t=2,\ldots,\tau). \tag{A-24}$$

Note that $\hat{x}_{1t}$ does not appear explicitly in this derivation.

It was stated in Section 3 that the linear form of the maintenance cost function yields a desirable property of the range of x_{3t}, viz., that its values are precisely multiples of the slope U_1 of the linear function. With $u_a = U_0 + U_1 a$, $y_\tau = \bar{x}_{2,t+1} - \bar{x}_{2\tau}$ is the number of purchases in year τ corresponding to the "purchase late" scenario.

Let z_τ be any feasible number of purchases in year τ, given $\hat{x}_{1t}$ and $\hat{x}_{2t}$. Then

$$x_{3t} - \lambda(x_{3t};\hat{x}_{1t},\hat{x}_{2t}) = \sum_{\tau=1}^{t-1} u_{t-\tau}z_\tau - \sum_{\tau=1}^{t-1} u_{t-\tau}y_\tau$$

$$= \sum_{\tau=1}^{t-1} [U_0 + U_1 (t-\tau)] (z_\tau - y_\tau) = U_1 [\sum_{\tau=1}^{t-1} \tau(z_\tau - y_\tau)],$$

where the last equality follows from the fact that

$$\sum_{\tau=1}^{t-1} y_\tau = \sum_{\tau=1}^{t-1} z_\tau = \hat{x}_{2t}.$$

Note that the number of values for x_{3t} is $\sum_{\tau=1}^{t-1} \tau (z_\tau - y_\tau) + 1$.

We turn now to establishing bounds on the decision variables. Assume fixed values of x_{1t} and x_{2t} in their appropriate ranges, say $\lambda(x_{1t}) \le \hat{x}_{1t} \le \mu(x_{1t})$ and $\lambda(x_{2t};\hat{x}_{1t}) \le \hat{x}_{2t} \le \mu(x_{2t})$; the state variable x_3 does not play a role. We know that $d_{1t} \ge 0$ from (A-3), and that $x_{1t} - d_{1t} \le \mu(x_{1,t+1})$. Thus

$$\lambda(d_{1t};\hat{x}_{1t}) = \max [0, \hat{x}_{1t} - \mu(x_{1,t+1})] \quad (t=1,\ldots,T). \tag{A-25}$$

(Note that $\lambda(d_{1t};\hat{x}_{1t})$ is not a function of $\hat{x}_{2t}$.)

Relations (A-3) also state that $d_{1t} \le N_t$, and we have $\hat{x}_{1t} - d_{1t} \ge \lambda(x_{1,t+1})$. In addition to these constraints, d_{1t} must be chosen so that the resulting $x_{1,t+1}$ yields a lower bound on $x_{2,t+1}$ that can be "reached" from $\hat{x}_{2t}$, i.e. $\hat{x}_{2t} + M_t \ge \lambda(x_{2,t+1};\hat{x}_{1t} - d_{1t})$. U ng the definition of $\lambda(x_{2,t+1};\hat{x}_{1t} - d_{1t})$, we have

$$\hat{x}_{2t} + M_t \geq \max_{1 \leq \tau \leq t+1} \{D_{\tau-1} - \sum_{j=t+1}^{\tau-1} M_j - \min [\mu(x_{1\tau}), \hat{x}_{1t} - d_{1t} + \sum_{j=\tau}^{t} N_j]\}$$

$$= \max \{ \max_{1 \leq \tau \leq T+1} [D_{\tau-1} - \sum_{j-t+1}^{\tau-1} M_j - \mu(x_{1\tau})], \tag{A-26}$$

$$\max_{1 \leq \tau \leq T+1} [D_{\tau-1} - \sum_{j=t+1}^{\tau-1} M_j - \hat{x}_{1t} + d_{1t} - \sum_{j=\tau}^{t} N_j]\}.$$

The part of (A-26) involving d_{1t} becomes

$$d_{1t} \leq \hat{x}_{1t} + \hat{x}_{2t} + M_t - \max_{1 \leq \tau \leq T+1} [D_{\tau-1} - \sum_{j=t+1}^{\tau-1} M_j - \sum_{j=\tau}^{t} N_j].$$

Therefore, we take

$$\mu(d_{1t};\hat{x}_{1t}, \hat{x}_{2t}) = \min \{N_t, \hat{x}_{1t} - \lambda(x_{1,t+1}),$$

$$\hat{x}_{1t} + \hat{x}_{2t} + M_t - \max_{1 \leq \tau \leq T+1} [D_{\tau-1} - \sum_{j=t+1}^{\tau-1} M_j - \sum_{j=\tau}^{t} N_j]\}. \tag{A-27}$$

That $\lambda(d_{1t};\hat{x}_{1t}, \hat{x}_{2t}) \leq \mu(d_{1t};\hat{x}_{1t}, \hat{x}_{2t})$ holds may be shown straightforwardly by taking $\hat{x}_{2t} = \lambda(x_{2t};\hat{x}_{1t})$ in the μ term and applying the definitions in (A-25) and (A-27).

Since we already have $\mu(d_{2t}) = M_t$ from (A-4), it remains only to find $\lambda(d_{2t};\hat{x}_{1t}, \hat{x}_{2t}, \hat{d}_{1t})$, where the three given variables fall in their respective ranges. Relations (A-4) state that $d_{2t} \geq 0$. In addition, we require $\hat{x}_{2t} + d_{2t} \geq \lambda(x_{2,t+1}; \hat{x}_{1t} - \hat{d}_{1t})$, so that

$$\lambda(d_{2t};\hat{x}_{1t},\hat{x}_{2t},\hat{d}_{1t}) = \max [0, \lambda(x_{2,t+1};\hat{x}_{1t} - \hat{d}_{1t}) - \hat{x}_{2t}] \ (t=1,\ldots T). \tag{A-28}$$

That $\lambda(d_{2t};\hat{x}_{1t},\hat{x}_{2t},\hat{d}_{1t}) \leq \mu(d_{2t}) = M_t$ holds also is straightforward to verify.

Observe that the ranges of d_{1t} and d_{2t}, developed above, do not depend on x_{3t}. In fact, we show below that the optimal decisions at any stage are independent of x_{3t}, because given $\hat{x}_{1t}$ and $\hat{x}_{2t}$, the value of the objective $f_t(x_t)$ at each stage is a linear function of x_{3t}, with the specific form

$$f_t(x_t) = g_t(x_{1t}, x_{2t}) + (\sum_{j=0}^{T-t} \delta^j)x_{3t}$$

where $\delta = 1/(1+r)$. For t=T, equations (3.3) and (3.4') imply

$$f_T(x_T) = \min_{d_T} I_T(x_T, d_T)$$

$$= g_T(x_{1T}, x_{2T}) + x_{3T},$$

with g_T taken as that part of (3.3) not involving x_{3T}. Now assuming that $f_t(x_t) = h_t(x_{1t}, x_{2t}) + (\sum_{j=0}^{T-t} \delta^j)\, x_{3t}$, we show that $f_{t-1}(x_{t-1}) = h_{t-1}(x_{1,t-1}, x_{2,t-1}) + (\sum_{j=0}^{T-t+1} \delta^j)\, x_{3,t-1}$, (i.e., "backwards induction" on t). We have

$$f_{t-1}(x_{t-1}) = \min_{d_{t-1}} [I_{t-1}(x_{t-1}, d_{t-1}) + \delta f_t(x_t)]$$

$$= \min_{d_{t-1}} [I_{t-1} + \delta h_t + \delta(\sum_{j=0}^{T-t} \delta^j)(x_{3,t-1} + u_1 d_{2,t-1} + U_1 x_{2,t-1})]$$

$$= \min_{d_{t-1}} [g_{t-1} + \delta h_t + (\sum_{j=1}^{T-t+1} \delta^j)(u_1 d_{2,t-1} + U_1 x_{2,t-1}) + (\sum_{j=1}^{T-t+1} \delta^j) x_{3,t-1} + x_{3,t-1}],$$

where g_{t-1} is that part of I_{t-1} not involving $x_{3,t-1}$ (cf. equation ( )).

$$f_{t-1}(x_{t-1}) = \min_{d_{t-1}} [h'_{t-1}(x_{1,t-1}, x_{2,t-1}) + (\sum_{j=0}^{T-t+1} \delta^j) x_{3,t-1}]$$

$$= h_{t-1}(x_{1,t-1}, x_{2,t-1}) + (\sum_{j=0}^{T-t+1} \delta^j) x_{3,t-1}$$

where $h'_{t-1} = g_{t-1} + \delta h_t + (\sum_{j=1}^{T-t+1} \delta^j)(u_1 d_{2,t-1} + U_1 x_{2,t-1})$.

The fact just proven makes it unnecessary to cycle through all of the values of d_{1t} and d_{2t} for each x_{3t}. We need only determine the optimal decisions for one value of x_{3t}, say $\lambda(x_{3t}; \hat{x}_{1t}, \hat{x}_{2t})$; these decisions are optimal for other values of x_{3t}, given $\hat{x}_{1t}$ and $\hat{x}_{2t}$, and the corresponding values of $f_t(x_t)$ may be calculated simply by adding the appropriate multiple of $(\sum_{j=0}^{T-t} \delta^j)$ to the optimal value of the objective function for $\lambda(x_{3t}; \hat{x}_{1t}; \hat{x}_{2t})$.

Table A-1 gives a summary of all formulas needed to calculate the ranges of the variables used in the dynamic programming model.

Table A-1 - FORMULAS FOR RANGES OF THE DYNAMIC PROGRAMMING MODEL VARIABLES

$$\lambda(x_{1t}) = \max \{0, \max_{1 \leq \tau \leq T+1} [D_{\tau-1} - \sum_{j=1}^{\tau-1} M_j - \sum_{j=\tau}^{t-1} N_j]\}$$

$$\mu(x_{1t}) = m - \max_{t \leq \tau \leq T+1} [n_\tau - \sum_{j=t}^{\tau-1} N_j]$$

$$\lambda(x_{2t}; \hat{x}_{1t}) = \max_{1 \leq \tau \leq T+1} \{D_{\tau-1} - \sum_{j=t}^{\tau-1} M_j - \min [\mu(x_{1\tau}), \hat{x}_{1t} + \sum_{j=\tau}^{t-1} N_j]\}$$

$$\mu(x_{2t}) = \sum_{j=1}^{t-1} M_j$$

$$\lambda(x_{3t}; \hat{x}_{1t}, \hat{x}_{2t}) = \sum_{\tau=1}^{t-1} U_{t-\tau} (\bar{x}_{2,\tau+1} - \bar{x}_{2\tau})$$

$$\bar{x}_{2\tau} = \max [x^*_{2\tau}, \hat{x}_{1t} - \sum_{j=\tau}^{t-1} M_j]$$

$$x^*_{2\tau} = \max_{1 \leq \sigma \leq t} [D_{\sigma-1} - x^*_{1\sigma} - \sum_{j=\tau}^{\sigma-1} M_j]$$

$$x^*_{1\sigma} = \min [\mu (x_{1\sigma}), \hat{x}_{1t} + \sum_{j=\sigma}^{t-1} N_j]$$

$$\mu(x_{3t}; \hat{x}_{1t}, \hat{x}_{2t}) = \sum_{\tau=1}^{t-1} U_{t-\tau}(\tilde{x}_{2,\tau+1} - \tilde{x}_{2\tau}).$$

$$\tilde{x}_{2\tau} = \sum_{j=1}^{\tau-1} M_j \text{ for } 1 \leq \tau < \sigma$$

$$\tilde{x}_{2\tau} = \hat{x}_{2t} \text{ for } \sigma \leq \tau \leq t.$$

σ is the <u>largest</u> value of k such that $\hat{x}_{2t} > \sum_{j=1}^{k-1} M_j$.

$$\lambda(d_{1t};\hat{x}_{1t}) = \max\,[0,\hat{x}_{1t} - \mu(x_{1,t+1})]$$

$$\mu(d_{1t};\hat{x}_{1t},\hat{x}_{2t}) = \min\,\{N_t,\hat{x}_{1t} - \lambda(x_{1,t+1}), x_{1t} + x_{2t} + M_t$$

$$- \max_{1\leq\tau\leq T+1}\,[D_{\tau-1} - \sum_{j=t+1}^{\tau-1} M_j - \sum_{j=\tau}^{t} N_j]\}$$

$$\lambda(d_{2t};\hat{x}_{1t},\hat{x}_{2t},\hat{d}_{1t}) = \max\,[0,\lambda(x_{2,t+1};\hat{x}_{1t} - \hat{d}_{1t}) - \hat{x}_{2t}].$$

$$\mu(d_{2t}) = M_t.$$

APPENDIX B

LISTING OF THE COMPUTER CODE FOR THE DYNAMIC PROGRAMMING MODEL

```
C
C     REPLAC IS A DYNAMIC PROGRAMMING CODE DESIGNED FOR AN EQUIPMENT
C     REPLACEMENT PROBLEM
C
      IMPLICIT INTEGER (A-H,O-Z)
      REAL RATE,DELTA,DELTB,DELTAM
C
C     NYRS IS THE MAXIMUM VALUE OF TT, I.E. THE MAXIMUM NUMBER OF YEARS
C     IN THE PLANNING HORIZON OR THE MAXIMUM NUMBER OF STAGES.
C     NPCS IS THE MAXIMUM VALUE OF MM, I.E. THE MAXIMUM NUMBER OF
C     RESOURCES IN THE INITIAL FLEET.
C     MAXAGE IS THE MAXIMUM VALUE OF R.
C     NPOS IS THE MAXIMUM NUMBER OF VALUES FOR X1T IN ANY STAGE T.
C     MPOS IS THE MAXIMUM NUMBER OF VALUES FOR X2T IN ANY STAGE T, I.E.
C     THE NUMBER OF VALUES OF X2T ASSOCIATED WITH A SINGLE VALUE OF X1T
C     SUMMED OVER ALL VALUES OF X1T FOR STAGE T.
C     STATE IS THE MAXIMUM NUMBER OF STATES PER STAGE.
C     NOX1 IS THE TOTAL NUMBER OF VALUES OF X1T IN ALL YEARS T, I.E. THE
C     SUM OF THE NUMBER OF VALUES OF X1T IN YEAR 1, THE NUMBER OF VALUES
C     IN YEAR 2, ETC.
C
      PARAMETER NYRS=25,NPCS=100,MAXAGE=25,STATE=10000
      PARAMETER NPOS=NPCS/4,MPOS=NPOS**2,NYRS1=NYRS+1,STATE2=2*STATE
      PARAMETER NOX1=NYRS1*NPCS
      COMMON M(NYRS),P(NYRS),LX1(NYRS1),MX1(NYRS1),LX2X1(NOX1),RN,DELTA,
     *       MX2(NYRS1),FTP1(STATE),IS(NPCS),D(NYRS),JNDX1(NPOS),U0,U1,
     *       JNDX2(MPOS),FLAG,RTP1,INDEX,V(MAXAGE,NYRS),NX1(NYRS1),TT,
     *       NN(NYRS),Q(MAXAGE),N(NYRS1),FT(STATE),INDX1(NPOS),
     *       INDX2(MPOS),DN(STATE2),MM,NOX11
      COMMON X1T,X2T,X3T,D1T,D2T,Y1TP1,Y2TP1,Y3TP1,T
      NOX11 = NOX1
      IOUT = 39
C
C     TT IS THE NUMBER OF YEARS IN THE PLANNING HORIZON (THE NUMBER OF
C     STAGES)
C     MM IS THE NUMBER OF RESOURCES INITIALLY ON HAND
C     R IS THE MAXIMUM ALLOWABLE AGE OF RESOURCES (RESOURCES OF AGE R
C     INITIALLY MUST BE RETIRED IN YEAR 1)
C
      READ (5,800) TT,MM,R
  800 FORMAT (3I5)
C     U0 AND U1 ARE COEFFICIENTS OF THE LINEAR MAINTENANCE FUNCTION.
C     MAINTENANCE COSTS ON A RESOURCE OF AGE A IS CALCULATED AS
```

```
00120    C      J0 + J1*A.  MAINTENANCE COSTS ARE IN PENNIES.
00120    C      BASE IS THE YEAR AROUND WHICH PURCHASE PRICES ARE BASED.
00120    C      P0 AND P1 ARE COEFFICIENTS OF THE PURCHASE PRICE FUNCTION.
00120    C      DEPRECIATION IS CALCULATED AS A GEOMETRIC DECREASE IN PURCHASE
00120    C      PRICE OVER R YEARS.  THE RATE OF DEPRECIATION IS (1 - RATE).
00120    C      PURCHASE PRICIES ARE IN PENNIES.
00120    C
00121           READ (5,805) J0,J1,BASE,P0,P1
00130       805 FORMAT (5I10)
00130    C
00130    C      M(T) IS THE MAXIMUM NUMBER OF PURCHASES ALLOWED IN  YEAR T
00130    C
00131           READ (5,810) (M(T),T=1,TT)
00137       810 FORMAT (16I5)
00137    C
00137    C      NN(T) IS THE MAXIMUM NUMBER OF RETIREMENTS ALLOWED IN YEAR T
00137    C
00140           READ (5,810) (NN(T),T=1,TT)
00140    C
00140    C      Q(I) IS THE NUMBER OF RESOURCES OF AGE I IN THE INITIAL FLEET
00140    C
00146           READ (5,810) (Q(I),I=1,R)
00146    C
00146    C      D(T) IS THE MINIMUM NUMBER OF RESOURCES REQUIRED IN YEAR T
00146    C
00154           READ (5,810) (D(T),T=1,TT)
00162           BASE = BASE-1901
00163           RATE =.6
00164           DELTA = .085
00164    C
00164    C      V(A,T) IS THE SALVAGE VALUE IN YEAR T OF A RESOURCE WHICH WAS
00164    C      INITIALLY OF AGE A
00164    C
00165           DO 2 A=1,R
00170           C=P0+P1*(BASE-A+1)
00171           DO 2 T=1,TT
00174           V(A,T) = 0
00175           IF (A+T-1 .LE. R) V(A,T) = FLOAT(C)*RATE**(A+T-1) + .5
00177         2 CONTINUE
00202           WRITE (6,989) ((V(A,T),T=1,TT),A=1,R)
00213       989 FORMAT (5I10)
00214           J=0
00214    C
00214    C      IS(J) IS THE AGE OF THE J-TH RESOURCE IN THE INITIAL FLEET.
00214    C      RESOURCE 1 IS YOUNGEST.
00214    C
00215           DO 5 I=1,R
00220           K=Q(I)
00221           IF (K .EQ. 0) GO TO 5
00223           DO 4 I1=1,K
00226           J=J+1
00227           IS(J) = I
00230         4 CONTINUE
00232         5 CONTINUE
00234           TTP1 = TT+1
00234    C
00234    C      N(T) IS THE NUMBER OF RESOURCES WHICH MUST BE RETIRED PRIOR TO
00234    C      YEAR T
```

```
C
      DO 30 T=1,TTP1
      IF (T .LT. TTP1) P(T) = P0+P1*(BASE+T)
      IF (T .GT. 1) GO TO 10
      N(1) = 0
      GO TO 30
   10 N(T) = N(T-1)+Q(R-T+2)
   30 CONTINUE
C
C     COMPUTE LX1(T) AND MX1(T), THE LOWER AND UPPER LIMITS FOR X1T.
C     X1T IS THE NUMBER OF RESOURCES REMAINING FROM THE INITIAL FLEET
C     IN YEAR T-1.
C
      DO 70 T=1,TTP1
      DO 60 TAU=1,TTP1
      IF (TAU .GT. 1) GO TO 35
      SUM = MM
      GO TO 45
   35 TAUM1 = TAU-1
      SUM = D(TAUM1)
      DO 40 J=1,TAUM1
      SUM = SUM-M(J)
   40 CONTINUE
   45 IF (TAU .GE. T) GO TO 55
      TM1 = T-1
      DO 50 J=TAU,TM1
      SUM = SUM-NN(J)
   50 CONTINUE
   55 IF (SUM .GT. LX1(T)) LX1(T) = SUM
   60 CONTINUE
      MX1(T) = MM-N(T)
      IF (T .EQ. TTP1) GO TO 69
      TP1 = T+1
      DO 64 TAU=TP1,TTP1
      SUM = N(TAU)
      TAUM1 = TAU-1
      DO 62 J=T,TAUM1
      SUM = SUM-NN(J)
   62 CONTINUE
      IF (MM-SUM .LT. MX1(T)) MX1(T) = MM- SUM
   64 CONTINUE
   69 IF (LX1(T) .LE. MX1(T)) GO TO 70
      WRITE (6,940) T,LX1(T),MX1(T)
  940 FORMAT (////' ERROR - THE PROBLEM IS INFEASIBLE.  FOR YEAR T = ',
     *I3,' LAMBDA(X1T) = ',I3,' IS GREATER THAN MU(X1T) = ',I3)
      STOP
   70 CONTINUE
      IF (LX1(1) .EQ. MM .AND. MX1(1) .EQ. MM) GO TO 71
      WRITE (6,945) LX1(1),MX1(1),MM
  945 FORMAT(////' ERROR - THE PROBLEM IS INFEASIBLE. FOR YEAR T = 1, LA
     *MBDA(X1T) = ',I3,' AND MU(X1T) = ',I3,' BUT THE INITIAL FLEET SIZE
     * IS MM = ',I3)
      STOP
C
C     SUBROUTINE X2LIM CALCULATES LIMITS ON X2T.
C     X2T IS THE NUMBER OF RESOURCES PURCHASED PRIOR TO YEAR T.
C     MX2(T) IS THE UPPER LIMIT OF X2T IN YEAR T.
C     IX1(T) IS THE NUMBER OF VALUES X1T ASSUMES THROUGH YEAR T.
```

```
00356   C      LX2X1(J) IS THE LOWER LIMIT OF X2T GIVEN THE J-TH VALUE OF X1T.
00356   C      LX1(1)=MX1(1)=MM IS THE FIRST VALUE OF X1T, LX1(2) THE SECOND,
00356   C      LX1(2)+1 THE THIRD, ETC.
00356   C
00357       71 CALL X2LIM
00360          DELTB = 1.0/(1.0+DELTA)
00361          DELTAM = 1.0
00361   C
00361   C      LOOP OF 180 IS THE BACKWARDS PASS OF THE DYNAMIC PROGRAM
00361   C
00362          DO 180 T=TT,1,-1
00365          IF (T .LT. TT) DELTAM=DELTAM+DELTB**(TT-T)
00367          ICOUNT = 0
00370          I1 = 0
00371          I2 = 0
00372          L1 = LX1(T)
00373          M1 = MX1(T)
00374          M2 = MX2(T)
00374   C
00374   C      122 BEGINS THE LOOP ON X1T
00374   C
00375      122 DO 140 X1T=L1,M1
00400          K=0
00401          IF (T .GT. 1) K=NX1(T-1)
00403          L2 = LX2X1(K+X1T-L1+1)
00404          I1 = I1+1
00405          IF (I1 .LE. NPOS) GO TO 123
00407          NPOS1 = NPOS
00410          WRITE (6,948) T,NPOS1
00414      948 FORMAT (////' ERROR - IN YEAR T = ',I3,' X1T ASSUMES MORE THAN NPO
00414         *S = ',I4,' VALUES. INCREASE THE VALUE OF NPOS ON THE PARAMETER CAR
00414         *DS'/' IN THE MAIN PROGRAM AND ALL SUBROUTINES.')
00415          STOP
00416      123 INDX1(I1) = M2-L2+1
00417          WRITE (6,997) L2
00422      997 FORMAT (' L2=',I10)
00423          DO 140 X2T=L2,M2
00426          I2 = I2+1
00427          L3 = 0
00430          M3 = 0
00431          IF (T .EQ. 1) GO TO 124
00433          FLAG = 0
00434          CALL X3LIM(T,X1T,X2T,L3,M3)
00435          IF (I2 .LE. MPOS) GO TO 124
00437          MPOS1 = MPOS
00440          WRITE (6,949) T,MPOS1
00444      949 FORMAT (////' ERROR - IN YEAR T = ',I3,' X2T ASSUMES MORE THAN MPO
00444         *S = ',I4,' VALUES. INCREASE THE VALUE OF MPOS ON THE PARAMETER CAR
00444         *DS'/' IN THE MAIN PROGRAM AND ALL SUBROUTINES.')
00445          STOP
00446      124 INDX2(I2) = (M3-L3)/U1 + 1
00447
00447   C      DETERMINE LIMITS ON D1T. D1T MUST BE LARGE ENOUGH SO THAT X1T-D1T
00447   C      IS LESS THAN OR EQUAL TO MX1(T+1) AND SMALL ENOUGH SO THAT THE
00447   C      LOWER LIMIT ON THE NUMBER OF RESOURCES PURCHASED PRIOR TO YEAR T+1
00447   C      IS LESS THAN OR EQUAL TO X2T+M(T)
00447   C
00447          MD1 = MIN(NN(T),X1T-LX1(T+1))
```

```
      BIGEST = -2**33
      DO 300 TAU=1,TTP1
      BIG = 0
      IF (TAU .GT. T) GO TO 296
      DO 295 J=TAU,T
      BIG = BIG-NN(J)
  295 CONTINUE
      IF (TAU .GT. 1) GO TO 296
      BIG = BIG+MM
      GO TO 298
  296 TAUM1 = TAU-1
      TP1 = T+1
      BIG = BIG+D(TAUM1)
      IF (TAU .LE. TP1) GO TO 298
      DO 297 J=TP1,TAUM1
      BIG = BIG-M(J)
  297 CONTINUE
  298 IF (BIG .GT. BIGEST) BIGEST=BIG
  300 CONTINUE
      MD1 = MIN(MD1,X1T+X2T+M(T)-BIGEST)
      LD1 = MAX(0,X1T-MX1(T+1))
      MD2 = M(T)
      WRITE (6,966) L3,M3,LD1,MD1,MD2
  966 FORMAT (' L3=',I10,' M3=',I10,' LD1=',I10,' MD1=',I10,' MD2=',I10)
      IF (LD1 .LE. MD1) GO TO 301
      WRITE (6,946) T,X1T,X2T,LD1,MD1
  946 FORMAT (////' ERROR - THE PROBLEM IS INFEASIBLE. FOR YEAR T = ',I3
     *,' WITH X1T = ',I3,' AND X2T = ',I3,' LAMBDA(D1T) = ',I3,' IS GREA
     *TER'/' THAN MU(D1T) = ',I3)
      STOP
  301 DO 140 X3T=L3,M3,J1
      ICOUNT = ICOUNT+1
      IF (ICOUNT .LE. STATE) GO TO 125
      STATE1 = STATE
      WRITE (6,999) T,STATE1
  999 FORMAT (////' ERROR - IN YEAR T = ',I3,' THERE ARE MORE THAN STATE
     * = ',I6,' STATES. INCREASE THE VALUE OF STATE ON THE PARAMETER CAR
     *DS'/' IN THE MAIN PROGRAM AND ALL SUBROUTINES.')
      STOP
  125 KINC = (X3T-L3)/J1
      IF (KINC .GT. 0) GO TO 132
      RTBEST = 2**35
      DO 130 D1T=LD1,MD1
      IX1TP1 = X1T-D1T-LX1(T+1)+NX1(T)+1
      LD2=MAX(0,LX2X1(IX1TP1)-X2T)
      IF (LD2 .LE. MD2) GO TO 127
      WRITE (6,947) T,X1T,X2T,D1T,LD2,MD2
  947 FORMAT (////' ERROR - THE PROBLEM IS INFEASIBLE. FOR YEAR T = ',I3
     *,' WITH X1T = ',I3,', X2T = ',I3,' AND D1T = ',I3,' LAMBDA(D2T) =
     *',I3,' IS'/' GREATER THAN MU(D2T) = ',I3)
      STOP
  127 DO 130 D2T=LD2,MD2
      CALL STGRET
      RTP1 = 0
      IF (T .LT. TT) CALL TRANFM
      RT = FLOAT(RN)+DELTR*FLOAT(RTP1)+.5
      WRITE (6,992) X1T,X2T,X3T,D1T,D2T,RN,RTP1,RT, TBEST,D1BEST,D2BEST
  992 FORMAT (2I5,I15,2I5,4I15,2I5)
```

```
  125 IF (RTBEST .LT. RT) GO TO 130
      RTBEST = RT
      D1BEST = D1T
      D2BEST = D2T
  130 CONTINUE
C
C     FT(ICOUNT) IS THE MINIMUM COST FROM STAGE T THROUGH TT, GIVEN THE
C     STATE IN YEAR T IS THE ICOUNT-TH STATE OF YEAR T.
C     DN(IC2-1) AND DN(IC2) ARE THE OPTIMAL DECISIONS IN STAGE T, GIVEN
C     THE ICOUNT-TH STATE IN STAGE T.
C
      FT(ICOUNT) = RTBEST
  132 IC2 = ICOUNT*2
      DN(IC2-1) = D1BEST
      DN(IC2) = D2BEST
      IF (KINC .GT. 0) FT(ICOUNT) = FLOAT(RTBEST)+DELTAM*FLOAT(KINC)*
     *FLOAT(U1)+.5
      WRITE (6,993) ICOUNT,X1T,X2T,X3T,DN(IC2-1),DN(IC2),FT(ICOUNT)
  993 FORMAT (3I5,I15,2I5,I15)
  140 CONTINUE
      SUM = 0
      NO = M1-L1+1
      DO 150 I=1,NO
      SUM = SUM + INDX1(I)
      JNDX1(I) = INDX1(I)
  150 CONTINUE
      WRITE (6,900) T,NO,ICOUNT
  900 FORMAT (3I5)
      WRITE (6,905) (INDX1(I),I=1,NO),SUM
  905 FORMAT (40I3)
      WRITE (6,905) (INDX2(I),I=1,SUM)
      WRITE (6,905) (DN(I),I=1,IC2)
      IF (T .EQ. 1) GO TO 155
      WRITE (IOUT) T,NO,ICOUNT
      WRITE (IOUT) (INDX1(I),I=1,NO),SUM
      WRITE (IOUT) (INDX2(I),I=1,SUM)
      WRITE (IOUT) (DN(I),I=1,IC2)
C
C     JNDX1(J) IS THE NUMBER OF VALUES OF X2T WHICH ARE COMPATIBLE WITH
C     THE J-TH VALUE OF X1T IN STAGE T+1.
C     JNDX2(K) IS THE NUMBER OF VALUES OF X3T WHICH ARE COMPATIBLE WITH
C     THE K-TH VALUE OF X2T.    -
C
  155 DO 160 I=1,SUM
      JNDX2(I) = INDX2(I)
  160 CONTINUE
      DO 170 I=1,ICOUNT
      FTP1(I) = FT(I)
  170 CONTINUE
  180 CONTINUE
      ENDFILE IOUT
      DO 200 I=1,4
      BACKSPACE IOUT
  200 CONTINUE
C
C     MM,0,0 IS THE INITIAL STATE IN YEAR 1.
C
      X1T = MM
```

```
      BIGEST = -2**33
      DO 300 TAU=1,ITP1
      BIG = 0
      IF (TAU .GT. T) GO TO 295
      DO 295 J=TAU,T
      BIG = BIG-NN(J)
  295 CONTINUE
      IF (TAU .GT. 1) GO TO 296
      BIG = BIG+MM
      GO TO 298
  296 TAUM1 = TAU-1
      TP1 = T+1
      BIG = BIG+D(TAUM1)
      IF (TAU .LE. TP1) GO TO 298
      DO 297 J=TP1,TAUM1
      BIG = BIG-M(J)
  297 CONTINUE
  298 IF (BIG .GT. BIGEST) BIGEST=BIG
  300 CONTINUE
      MD1 = MIN(MD1,X1T+X2T+M(T)-BIGEST)
      LD1 = MAX(0,X1T-MX1(T+1))
      MD2 = M(T)
      WRITE (6,966) L3,M3,LD1,MD1,MD2
  966 FORMAT (' L3=',I10,' M3=',I10,' LD1=',I10,' MD1=',I10,' MD2=',I10)
      IF (LD1 .LE. MD1) GO TO 301
      WRITE (6,946) T,X1T,X2T,LD1,MD1
  946 FORMAT (////' ERROR - THE PROBLEM IS INFEASIBLE. FOR YEAR T = ',I3
     *,' WITH X1T = ',I3,' AND X2T = ',I3,' LAMBDA(D1T) = ',I3,' IS GREA
     *TER'/' THAN MU(D1T) = ',I3)
      STOP
  301 DO 140 X3T=L3,M3,J1
      ICOUNT = ICOUNT+1
      IF (ICOUNT .LE. STATE) GO TO 125
      STATE1 = STATE
      WRITE (6,999) T,STATE1
  999 FORMAT (////' ERROR - IN YEAR T = ',I3,' THERE ARE MORE THAN STATE
     * = ',I6,' STATES. INCREASE THE VALUE OF STATE ON THE PARAMETER CAR
     *DS'/' IN THE MAIN PROGRAM AND ALL SUBROUTINES.')
      STOP
  125 KINC = (X3T-L3)/J1
      IF (KINC .GT. 0) GO TO 132
      RTBEST = 2**33
      DO 130 D1T=LD1,MD1
      IX1TP1 = X1T-D1T-LX1(T+1)+NX1(T)+1
      LD2=MAX(0,LX2X1(IX1TP1)-X2T)
      IF (LD2 .LE. MD2) GO TO 127
      WRITE (6,947) T,X1T,X2T,D1T,LD2,MD2
  947 FORMAT (////' ERROR - THE PROBLEM IS INFEASIBLE. FOR YEAR T = ',I3
     *,' WITH X1T = ',I3,', X2T = ',I3,' AND D1T = ',I3,' LAMBDA(D2T) =
     *',I3,' IS'/' GREATER THAN MU(D2T) = ',I3)
      STOP
  127 DO 130 D2T=LD2,MD2
      CALL STGRET
      RTP1 = 0
      IF (T .LT. II) CALL TRANEM
      RT = FLOAT(RN)+DELTR*FLOAT(RTP1)+.5
      WRITE (6,992) X1T,X2T,X3T,D1T,D2T,RN,RTP1,RT, T3EST,D1BEST,D2BEST
  992 FORMAT (2I5,I15,2I5,4I15,2I5)
```

```
00626        125 IF (RTBEST .LT. RT) GO TO 130
00630            RTBEST = RT
00631            D1BEST = D1T
00632            D2BEST = D2T
00633        130 CONTINUE
00633     C
00633     C      FT(ICOUNT) IS THE MINIMUM COST FROM STAGE T THROUGH TT, GIVEN THE
00633     C      STATE IN YEAR T IS THE ICOUNT-TH STATE OF YEAR T.
00633     C      DN(IC2-1) AND DN(IC2) ARE THE OPTIMAL DECISIONS IN STAGE T, GIVEN
00633     C      THE ICOUNT-TH STATE IN STAGE T.
00633     C
00636            FT(ICOUNT) = RTBEST
00637        132 IC2 = ICOUNT*2
00640            DN(IC2-1) = D1BEST
00641            DN(IC2) = D2BEST
00642            IF (KINC .GT. 0) FT(ICOUNT) = FLOAT(RTBEST)+DELTAM*FLOAT(KINC)*
00642           *FLOAT(J1)+.5
00644            WRITE (6,993) ICOUNT,X1T,X2T,X3T,DN(IC2-1),DN(IC2),FT(ICOUNT)
00655        993 FORMAT (3I5,I15,2I5,I15)
00656        140 CONTINUE
00662            SUM = 0
00663            NO = M1-L1+1
00664            DO 150 I=1,NO
00667            SUM = SUM + INDX1(I)
00670            UNDX1(I) = INDX1(I)
00671        150 CONTINUE
00673            WRITE (6,900) T,NO,ICOUNT
00700        900 FORMAT (3I5)
00701            WRITE (6,905) (INDX1(I),I=1,NO),SUM
00710        905 FORMAT (40I3)
00711            WRITE (6,905) (INDX2(I),I=1,SUM)
00717            WRITE (6,905) (DN(I),I=1,IC2)
00725            IF (T .EQ. 1) GO TO 155
00727            WRITE (IOUT) T,NO,ICOUNT
00734            WRITE (IOUT) (INDX1(I),I=1,NO),SUM
00743            WRITE (IOUT) (INDX2(I),I=1,SUM)
00751            WRITE (IOUT) (DN(I),I=1,IC2)
00751     C
00751     C      UNDX1(J) IS THE NUMBER OF VALUES OF X2T WHICH ARE COMPATIBLE WITH
00751     C      THE J-TH VALUE OF X1T IN STAGE T+1.
00751     C      UNDX2(K) IS THE NUMBER OF VALUES OF X3T WHICH ARE COMPATIPLE WITH
00751     C      THE K-TH VALUE OF X2T.
00751     C
00757        155 DO 160 I=1,SUM
00762            UNDX2(I) = INDX2(I)
00763        160 CONTINUE
00765            DO 170 I=1,ICOUNT
00770            FTP1(I) = FT(I)
00771        170 CONTINUE
00773        180 CONTINUE
00775            ENDFILE IOUT
00776            DO 200 I=1,4
01001            BACKSPACE IOUT
01002        200 CONTINUE
01002     C
01002     C      MM,0,0 IS THE INITIAL STATE IN YEAR 1.
01002     C
01004            X1T = MM
```

```
01005          X2T = 0
01006          X3T = 0
01007          FT(1) = FT(1)/100
01010          WRITE (6,950) FT(1)
01013      950 FORMAT ('1THE TOTAL COST OF THE FOLLOWING EQUIPMENT REPLACEMENT PO
01013         *LICY IS  $',I15///' YEAR      SELL       BUY ')
01014          WRITE (6,955) T,DN(1),DN(2)
01021      955 FORMAT (I4,I9,I10)
01022          D1T = DN(1)
01023          D2T = DN(2)
01024          TTM1 = TT-1
01024    C
01024    C     THE FORWARD PASS OF THE DYNAMIC PROGRAM BEGINS HERE.
01024    C
01025          DO 220 T=1,TTM1
01030          READ (IOUT) TP1,NO,ICOUNT
01035          READ (IOUT) (UNDX1(I),I=1,NO),SUM
01044          READ (IOUT) (UNDX2(I),I=1,SUM)
01052          IC2 = 2*ICOUNT
01053          READ (IOUT) (DN(I),I=1,IC2)
01061          CALL TRANFM
01062          D1T = DN(INDEX*2-1)
01063          D2T = DN(INDEX*2)
01064          X1T = Y1TP1
01065          X2T = Y2TP1
01066          X3T = Y3TP1
01067          WRITE (6,955) TP1,D1T,D2T
01074          IF (T .EQ. TTM1) STOP
01076          DO 205 I=1,8
01101          BACKSPACE IOUT
01102      205 CONTINUE
01104      220 CONTINUE
01106          END
00101          SUBROUTINE STGRET
00101    C
00101    C     STGRET CALCULATES THE STAGE RETURN.
00101    C
00103          IMPLICIT INTEGER (A-H,O-Z)
00104          REAL DELTA
00105          PARAMETER NYRS=25,NPCS=100,MAXAGE=25,STATE=10000
00106          PARAMETER NPOS=NPCS/4,MPOS=NPOS**2,NYRS1=NYRS+1,STATE2=2*STATE
00107          PARAMETER NOX1=NYRS1*NPCS
00110          COMMON M(NYRS),P(NYRS),LX1(NYRS1),MX1(NYRS1),LX2X1(NOX1),RN,DELTA,
00110         *        MX2(NYRS1),FTP1(STATE),IS(NPCS),D(NYRS),UNDX1(NPOS),U0,U1,
00110         *        UNDX2(MPOS),FLAG,RTP1,INDEX,V(MAXAGE,NYRS),NX1(NYRS1),TT,
00110         *        NN(NYRS),Q(MAXAGE),N(NYRS1),FT(STATE),INDX1(NPOS),
00110         *        INDX2(MPOS),DN(STATE2),MM,NOX11
00111          COMMON X1T,X2T,X3T,D1T,D2T,Y1TP1,Y2TP1,Y3TP1,T
00112          RN=0
00113          LEFT=X1T-D1T
00114          IF (LEFT.LT.1) GO TO 20
00116          DO 10 I=1,LEFT
00121          RN = RN+U0+(IS(I)+T)*U1
00122       10 CONTINUE

00124       20 IF (D1T.LT.1) GO TO 40
00126          DO 30 I=1,D1T
00131          J=IS(I+LEFT)
00132          RN = RN-V(J,T)
00133       30 CONTINUE
00135       40 IF (D2T.GT.0) RN=RN+(P(T)+U0+U1)*D2T
00137          IF (X2T.GT.0) RN=RN+X3T+U1*X2T
00141          RETURN
00142          END
```

```
      SUBROUTINE TRANFM
C
C     TRANFM USES THE STAGE TRANSFORMATION TO PICK UP THE CORRECT VALUE
C     OF THE MINIMAL COST FROM STAGE T+1 TO TT, RTP1.
C
      IMPLICIT INTEGER (A-H,O-Z)
      REAL DELTA
      PARAMETER NYRS=25,NPCS=100,MAXAGE=25,STATE=10000
      PARAMETER NPOS=NPCS/4,MPOS=NPOS**2,NYRS1=NYRS+1,STATE2=2*STATE
      PARAMETER NOX1=NYRS1*NPCS
      COMMON M(NYRS),P(NYRS),LX1(NYRS1),MX1(NYRS1),LX2X1(NOX1),RN,DELTA,
     *       MX2(NYRS1),FTP1(STATE),IS(NPCS),D(NYRS),JNDX1(NPOS),J0,U1,
     *       JNDX2(MPOS),FLAG,RTP1,INDEX,V(MAXAGE,NYRS),NX1(NYRS1),TT,
     *       NN(NYRS),Q(MAXAGE),N(NYRS1),FT(STATE),INDX1(NPOS),
     *       INDX2(MPOS),DN(STATE2),MM,NOX11
      COMMON X1T,X2T,X3T,D1T,D2T,Y1TP1,Y2TP1,Y3TP1,T
      INDEX = 0
      Y1TP1 = X1T-D1T
      Y2TP1 = X2T+D2T

      Y3TP1=X3T+U1*X2T+(J0+U1)*D2T
      I1 = Y1TP1-LX1(T+1)
      I2=0
      IF (I1.LT.1) GO TO 20
      I3=0
      DO 10 J=1,I1
      I2=I3+1
      I3=JNDX1(J)+I3
      DO 10 K=I2,I3
      INDEX = INDEX+JNDX2(K)
   10 CONTINUE
   20 L2 = NX1(T)+Y1TP1-LX1(T+1)+1
      L2 = LX2X1(L2)
      I2=Y2TP1-L2
      IF (I2.LT.1) GO TO 40
      I4=I3+1
      I3=I3+I2
      DO 30 J=I4,I3
      INDEX=INDEX+JNDX2(J)
   30 CONTINUE
   40 TP1=T+1
      FLAG=1
      CALL X3LIM(TP1,Y1TP1,Y2TP1,L3,M3)
      INDEX = INDEX+(Y3TP1-L3)/U1+1
      RTP1=FTP1(INDEX)
      RETURN
      END
      SUBROUTINE X3LIM (T,X1HAT,X2HAT,L3,M3)
C
C     X3LIM CALCULATES BOUNDS L3 AND M3 ON X3T IN STAGE T, GIVEN X2T.
C
      IMPLICIT INTEGER (A-H,O-Z)
      REAL DELTA
      PARAMETER NYRS=25,NPCS=100,MAXAGE=25,STATE=10000
      PARAMETER NPOS=NPCS/4,MPOS=NPOS**2,NYRS1=NYRS+1,STATE2=2*STATE
      PARAMETER NOX1=NYRS1*NPCS
      COMMON M(NYRS),P(NYRS),LX1(NYRS1),MX1(NYRS1),LX2X1(NOX1),RN,DELTA,
     *       MX2(NYRS1),FTP1(STATE),IS(NPCS),D(NYRS),JNDX1(NPOS),J0,U1,
     *       JNDX2(MPOS),FLAG,RTP1,INDEX,V(MAXAGE,NYRS),NX1(NYRS1),TT,
     *       NN(NYRS),Q(MAXAGE),N(NYRS1),FT(STATE),INDX1(NPOS),
     *       INDX2(MPOS),DN(STATE2),MM,NOX11
      DIMENSION X23AR(NYRS1),X1STAR(NYRS1),X2STAR(NYRS1),X2TILD(NYRS1)
```

```
      L3 = 0
      TM1 = T-1
      DO 30 TAU=1,T
      IF (TAU .GT. 1) GO TO 20
      SUMM = 0
      DO 10 J=1,TM1
      SUMM = SUMM+NN(J)
   10 CONTINUE
   20 IF (TAU .GT. 1) SUMM = SUMM-NN(TAU-1)
      X1STAR(TAU) = MIN(MX1(TAU),X1HAT+SUMM)
   30 CONTINUE
      DO 90 TAU=1,T
      X2STAR(TAU) = -2**33
      DO 60 SIGMA = 1,T
      BIG = AM-X1STAR(SIGMA)
      IF (SIGMA .GT. 1) BIG = D(SIGMA-1)-X1STAR(SIGMA)
      IF (SIGMA .LE. TAU) GO TO 50
      SIGM1 = SIGMA-1
      DO 40 J=TAU,SIGM1
      BIG = BIG-M(J)
   40 CONTINUE
   50 IF (BIG .GT. X2STAR(TAU)) X2STAR(TAU) = BIG
   60 CONTINUE
      IF (TAU .GT. 1) GO TO 80
      SUMM = X2HAT
      DO 70 J=1,TM1
      SUMM = SUMM-M(J)
   70 CONTINUE
   80 IF (TAU .GT. 1) SUMM = SUMM+M(TAU-1)
      X2BAR(TAU) = MAX(X2STAR(TAU),SUMM)
   90 CONTINUE
      DO 100 TAU=1,TM1
      L3 = L3 + (U0+U1*(T-TAU))*(X2BAR(TAU+1)-X2BAR(TAU))
  100 CONTINUE
      IF (FLAG .EQ. 1) RETURN
      IF (T .EQ. 2) WRITE (6,900) T,X1HAT,X2HAT,(X1STAR(J),X2STAR(J),
     *X2BAR(J),J=1,T),L3
  900 FORMAT (9I5,I15)
      SUMM = 0
      DO 110 TAU=1,T
      IF (TAU .GT. 1) SUMM=SUMM+M(TAU-1)
      X2TILD(TAU) = X2HAT
      IF (X2HAT .GE. SUMM) X2TILD(TAU) = SUMM
  110 CONTINUE
      M3 = 0
      DO 120 TAU=1,TM1
      M3 = M3 + (U0+U1*(T-TAU))*(X2TILD(TAU+1)-X2TILD(TAU))
  120 CONTINUE
      IF (L3 .LE. M3) RETURN
      WRITE (6,905) T,X1HAT,X2HAT,L3,M3
  905 FORMAT (////' ERROR - THE PROBLEM IS INFEASIBLE. FOR YEAR T = ',I3
     *,' WITH X1T = ',I3,' AND X2T = ',I3,' LAMBDA(X3T) = ',I12,' IS GRE
     *ATER'/' THAN M(X3T) = ',I12)
      STOP
      END
```

```
      SUBROUTINE X2LIM
C
C     X2LIM CALCULATES BOUNDS ON X2T GIVEN EVERY POSSIBLE VALUE OF X1T
C     FOR ALL STAGES.
C
      IMPLICIT INTEGER (A-H,O-Z)
      REAL DELTA
      PARAMETER NYRS=25,NPCS=100,MAXAGE=25,STATE=10000
      PARAMETER NPOS=NPCS/4,MPOS=NPOS**2,NYRS1=NYRS+1,STATE2=2*STATE
      PARAMETER NOX1=NYRS1*NPCS
      COMMON M(NYRS),P(NYRS),LX1(NYRS1),MX1(NYRS1),LX2X1(NOX1),RM,DELTA,
     *       MX2(NYRS1),FTP1(STATE),IS(NPCS),D(NYRS),UNDX1(NPOS),J0,U1,
     *       UNDX2(MPOS),FLAG,RTP1,INDEX,V(MAXAGE,NYRS),NX1(NYRS1),TT,
     *       NN(NYRS),Q(MAXAGE),N(NYRS1),FT(STATE),INDX1(NPOS),

     *       INDX2(MPOS),ON(STATE2),MM,NOX11
      COMMON X1T,X2T,X3T,D1T,D2T,Y1TP1,Y2TP1,Y3TP1,T
      TTP1 = TT+1
      ICOUNT = 0
      MX2(1) = 0
      DO 50 T=1,TTP1
      IF (T .GT. 1) MX2(T) = MX2(T-1)+M(T-1)
      L1 = LX1(T)
      M1 = MX1(T)
      DO 40 X1T=L1,M1
      LAM1 = 0
      DO 30 TAU=2,TTP1
      SMALL = X1T
      IF (TAU .GE. T) GO TO 15
      TM1 = T-1
      DO 10 J=TAU,TM1
      SMALL = SMALL+NN(J)
   10 CONTINUE
   15 TLAM1 = D(TAU-1)-MIN(MX1(TAU),SMALL)
      IF (TAU .LE. T) GO TO 25
      TAUM1 = TAU-1
      DO 20 J=T,TAUM1
      TLAM1 = TLAM1-M(J)
   20 CONTINUE
   25 IF (TLAM1 .GT. LAM1) LAM1 = TLAM1
   30 CONTINUE
      IF (LAM1 .LE. MX2(T)) GO TO 38
      WRITE (6,900) T,X1T,MX2(T),LAM1
  900 FORMAT (/////' PROBLEM IS INFEASIBLE.   FOR T = ',I2,' AND X1T = '
     *,I3,' MU(X2T) = ',I3,' AND LAMBDA(X2T) = ',I3)
      STOP
   38 ICOUNT = ICOUNT+1
      IF (ICOUNT .LE. NOX11) GO TO 39
      WRITE (6,910)
  910 FORMAT (////' ERROR - X1T ASSUMES MORE THAN NOX1 VALUES. INCREASE
     *THE VALUE OF NOX1 ON THE PARAMETER CARDS IN THE MAIN PROGRAM AND'/
     *' ALL SUBROUTINES.')
      STOP
   39 LX2X1(ICOUNT) = LAM1
   40 CONTINUE
      NX1(T) = ICOUNT
   50 CONTINUE
      I=NX1(TTP1)
      WRITE (6,905) (LX2X1(J),J=1,I)
  905 FORMAT (25I5)
      END
```

APPENDIX C

AN INTEGER PROGRAMMING MODEL

The model described in this Appendix is a somewhat simplified integer programming (IP) analog to the DP model presented in Section 3. Since the IP version is subsumed under the DP version, the former is documented here for its own sake, as an application of integer programming, and is not necessarily intended to serve as an "alternative" model.

As in the DP model, the IP model prescribes actions to be taken each year for a T-year period to minimize the total cost over those T years. From a given initial fleet, the decisions specify the number of purchases each year and the number of retirements, from the initial fleet, of engines of each age $\underline{a}$. (Note that T may not be taken so large as to make liable to retirement engines which were purchased during the T-year period.) These decisions are to be made so as to minimize the total cost for the T years, subject to the constraint that a specified minimum fleet size be met each year.

The variables are:

x_{at} = the number of engines, initially of age $\underline{a}$, retired in year t,

y_t = the number of new engines purchased in year t.

The conventions regarding age definition and decision times are the same as for the DP model (cf., footnote 6).

The data required by the model include:

D_t = the minimum number of engines required during year t (checked against the fleet size after year t's decisions have been made),

M_t = the maximum number of engines which may be purchased in year t,

P_t = the purchase price of an engine in year t,

Q_a = the number of a - year - old engines in the initial fleet,

u_a = the maintenance cost of an engine during its a^{th} year of service,

v_{at} = the resale value in year t of an engine which was initially of age a.

Note that this model does not have a ceiling on the number of engines that may be retired, nor does it have a mandatory retirement age, as does the DP model. If a set A of ages of engines in the initial fleet is given, then the model requires data for u and v for a as large as $\mu + T$, where μ is the maximum age in A.

Using the above definitions, the IP is formulated as:

minimize

$$\sum_{t=1}^{T} \{(P_t + u_1)y_t + \sum_{a\varepsilon A} [u_{a+t} (Q_a - \sum_{\tau\leq t} x_{at}) - v_{at}x_{at}] + \sum_{\tau<t} u_{t-\tau+1}\, y_\tau\}^{14} \qquad \text{(C-1)}$$

[14] The term $\sum_{t=1}^{T} \sum_{a\varepsilon A} u_{a+t}Q_a$ in the objective function (C-1) does not affect the minimizing values of x_{at} and y_t, but it must be included to calculate the minimum value of (C-1). Also, discounting has been omitted for simplicity and could clearly be implemented in the model.

subject to

$$\sum_{t=1}^{T} x_{at} \leq Q_a \quad (a\varepsilon A), \tag{C-2}$$

$$y_t \leq M_t \quad (t=1,\ldots,T), \tag{C-3}$$

$$\sum_{a\varepsilon A} Q_a + \sum_{\tau\leq t} (y_\tau - \sum_{a\varepsilon A} x_{a\tau}) \geq D_t \quad (t=1,\ldots,T) \tag{C-4}$$

$$x_{at},\ y_t \text{ nonnegative integers } (a\varepsilon A,\ t=1,\ldots,T) \tag{C-5}$$

The expressions { } summed in (C-1) are the costs for the individual years t. Each of these is calculated from the following components:

$(P_t + u_1)$ = the cost of purchasing an engine and maintaining it during its first year of service,

$u_{a+t}(Q_a - \sum_{\tau\leq t} x_{at})$ = the maintenance cost in year t of engines, initially of age a, which remain in the fleet,

$v_{at}x_{at}$ = the revenue from retiring x_{at} engines, initially of age a, in year t,

$\sum_{\tau<t} u_{t-\tau+1}y_\tau$ = the maintenance cost in year t of engines purchased during years $\tau = 1, \ldots, t-1$.

Constraint (C-2) specifies that the total number of engines retired, initially of age a, not exceed the initial number of age a engines, and constraint (C-3) restricts to at most M_t the number of engines purchased in year t. Constraint (C-4) requires that the number of engines in the fleet in year t (after purchases and retirements

in year t) to be at least D_t. If d is the number of distinct ages in the set A of ages, then the IP in (C-1) through (C-5) has d + 2T constraints and (d + 1)T variables.

The reader may have observed that the IP described above does not specify any retirement order. The condition that engines be retired in order of decreasing age may be imposed by the following suggestion of A. J. Goldman. This uses (d - 1)T additional variables and 2(d - 1)T additional constraints :

$$\sum_{\alpha<a} x_{\alpha t} \leq (\sum_{\alpha<a} Q_\alpha)\, \delta_{at}, \tag{C-6}$$

$$Q_a - \sum_{\tau \leq t} x_{a\tau} \leq Q_a\, (1 - \delta_{at}), \tag{C-7}$$

with $a \varepsilon A$, $a \neq \min\{\alpha | \alpha \varepsilon A\}$ and $t = 1,\ldots,T$. The 0-1 variable δ_{at} acts as a "switch": if $\delta_{at} = 0$, then the retiring of engines of initial age less than a in year t is prohibited by (C-6), and (C-7) is non-constraining, whereas if $\delta_{at} = 1$ such engines may be retired since (C-7) together with (C-2) would imply that all Q_a engines, initially of age a, have been retired, and the right side of (C-6) is non-constraining in view of (C-2). Of course, the constraints (C-6) and (C-7) may be introduced only as they are needed. Thus if the IP (C-1) - (C-5) yields a solution in which retirements are partially "out of order," (C-6) and (C-7) would be imposed only for the exceptional pairs (a, t). The nature the solution will depend on the data, and if these are "reasonable" one might expect the "order"

condition to hold on its own.